For the other observer, the length is the spatial separation is along a line of simultaneity for the primed observer

$$L' = x'_3 - x'_1 = \gamma((x_3 - x_1) - \beta c(t_3 - t_1)) = 1.25(1.0 - 0.6(0.6)) = 0.8 = \frac{1}{\gamma} = \frac{L}{\gamma}$$

What is the spatial separation between events 1 and 2 for the primed observer? Does it have any physical meaning?

Time Dilation

Suppose the clock is at rest in the primed frame. Then the relevant events representing on the worldlines of the ends of an object are

$$(x'_1, ct'_1) = (0.0, 0.0) \text{ and } (x'_2, ct'_2) = (0.0, 1.0)$$ for primed observer

Then for unprimed observer we have

$$\Delta x = \gamma(\Delta x' + \beta c\Delta t') = \gamma\beta \quad , \quad c\Delta t = \gamma(c\Delta t' + \beta\Delta x') = \gamma \rightarrow \text{time dilation}$$

Note the change in signs in the Lorentz transformations when we go from the primed to the unprimed coordinates. Why?

Let us now return to the k-factor. Our original **k-factor** assumption says that, if the unprimed observer is sending out signals every T seconds and the primed observer is receiving them every T' seconds where $T' = kT$, then we have the relationship

$$f' = \frac{1}{T'} = \frac{1}{kT} = \frac{1}{k} f = \sqrt{\frac{c-v}{c+v}} f$$

between the frequency f as measured in the unprimed frame and the frequency f' as measured in the primed frame. The case above corresponds to the two observers moving away from each other. In this case $f' < f$ and hence the primed observer sees the wavelength increase (wavelength = $\lambda = c/f$), which is the famous **"red shift"**.

If they move towards each other, then $v \rightarrow -v$ or $k \rightarrow 1/k$ and the frequency increases (wavelength decreases) and we get a **"blue shift"**.

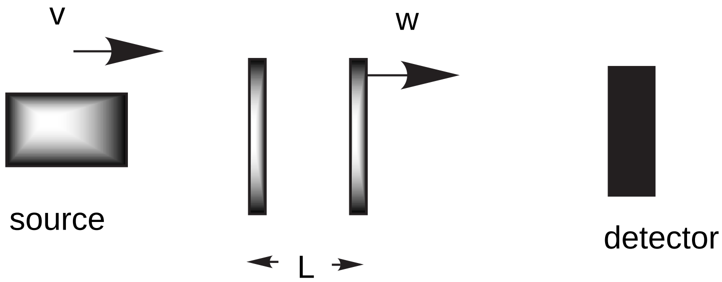
This is called the relativistic **Doppler effect** for light.

Let us look at the important Doppler effect in more detail.

The Doppler Effect

Sound and the Acoustic Doppler Effect

Sound travels through a medium such as air with a speed w. This speed is determined by the properties of the medium and is independent of the motion of the source. We consider a source of sound that is moving with velocity v through the medium towards an observer at rest. We assume for simplicity that the observer (detector) lies along the line of motion of the source.



As shown in the diagram, we represent the sound wave as a regular series of pulses. These pulses are separated in time by an amount $\tau_0 = \frac{1}{f_0}$ where f_0 is the frequency of the sound from the source.

In a time T the sound travels a distance wT, and if the pulses are separated by a distance L, the number reaching the detector is $\frac{wT}{L}$. The rate at which pulses arrive is

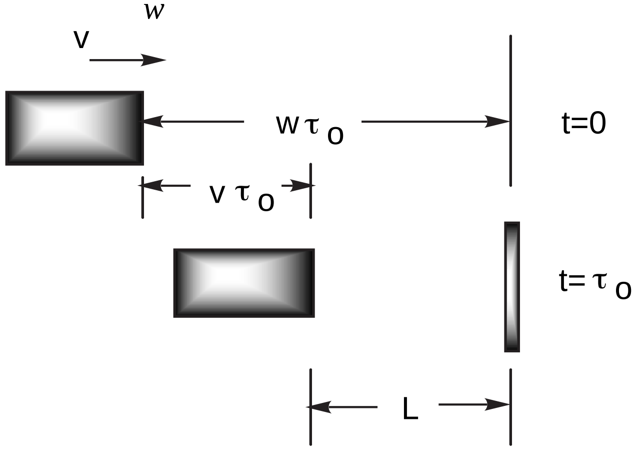
$$\frac{w}{L} = \text{frequency } \left(\frac{\text{number}}{\text{T}}\right) \text{ of sound at the detector } = f_{\text{D}}$$

To determine L, we consider a pulse emitted at $t = 0$ and a second pulse emitted at $t = \tau_0$. During the interval τ_0 the first pulse travels a distance $w\tau_0$ in the medium and the source travels a distance $v\tau_0$. As shown in the figure below the distance between the pulses is given by

$$L = w\tau_0 - v\tau_0 = (w - v)\tau_0 = \frac{(w - v)}{f_0}$$

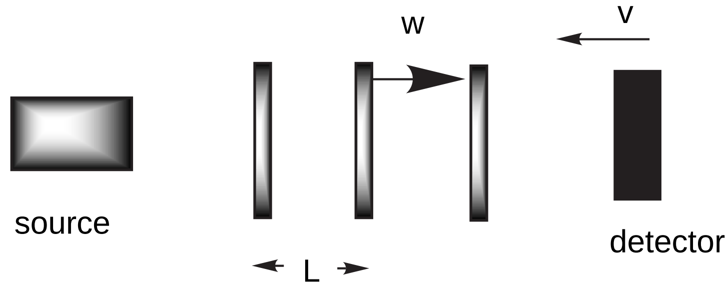
and

$$f_D = f_0 \frac{1}{1 - \frac{v}{w}} \qquad \text{for a moving source}$$



For an approaching source, $v > 0$ and thus $f_D > f_0$. For a receding source, $v < 0$ and thus $f_D < f_0$.

If the source is at rest and the detector is moving (as shown below) the situation is different.



The speed of the pulses relative to the detector is $w + v$. The rate at which the pulses arrive is

$$f_D = \frac{w+v}{L}$$

Since the source is at rest, $L = w\tau_0 = \frac{w}{f_0}$ and thus

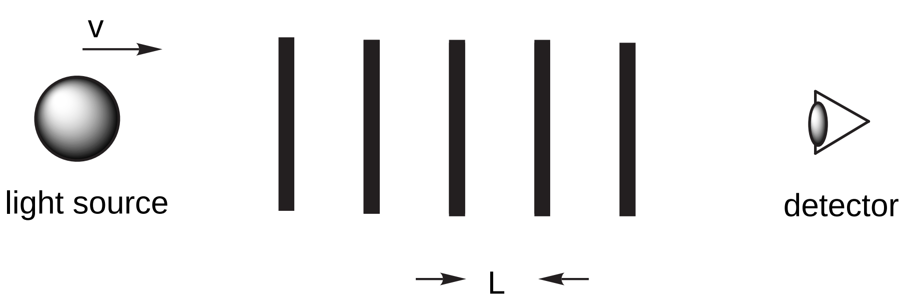
$$f_D = f_0(1 + \frac{v}{w}) \qquad \text{for a moving detector}$$

The two results are not **symmetric**. They are approximately the same for small $\frac{v}{w}$ since $\frac{1}{1 - \frac{v}{w}} \approx 1 + \frac{v}{w}$ in that case. If we know f_D, then we can tell whether it is the source or the detector that is moving!!

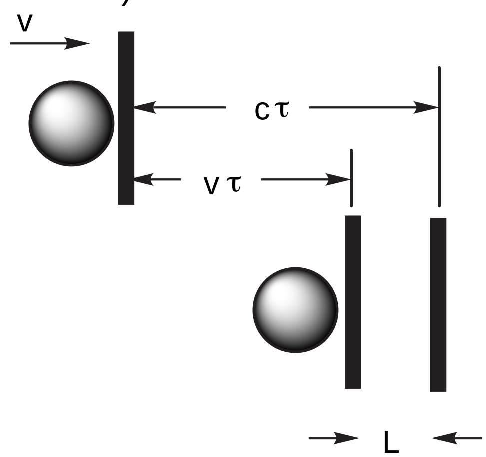
This is so because the speed of sound is not a universal constant, but only has a definite value relative to the medium where it is propagating.

Light and the Relativistic Doppler Effect

Suppose a light source flashes with period $\tau_0 = \frac{1}{f_0}$ in its rest frame and that the source is moving towards the observer(detector) with velocity v.



Due to time dilation, the period in the detector rest frame is $\tau = \gamma\tau_0$. Since the speed of light is a universal constant, the pulses arrive at the detector with speed c. As shown in the diagram below the frequency of the pulses is $f_D = \frac{c}{L}$, where L is the pulse separation in the detector frame. Since the source is moving towards the detector we have (see diagram below)



$$L = c\tau - v\tau = (c - v)\tau = (c - v)\gamma\tau_0 = \gamma\,\frac{(c - v)}{f_0}$$

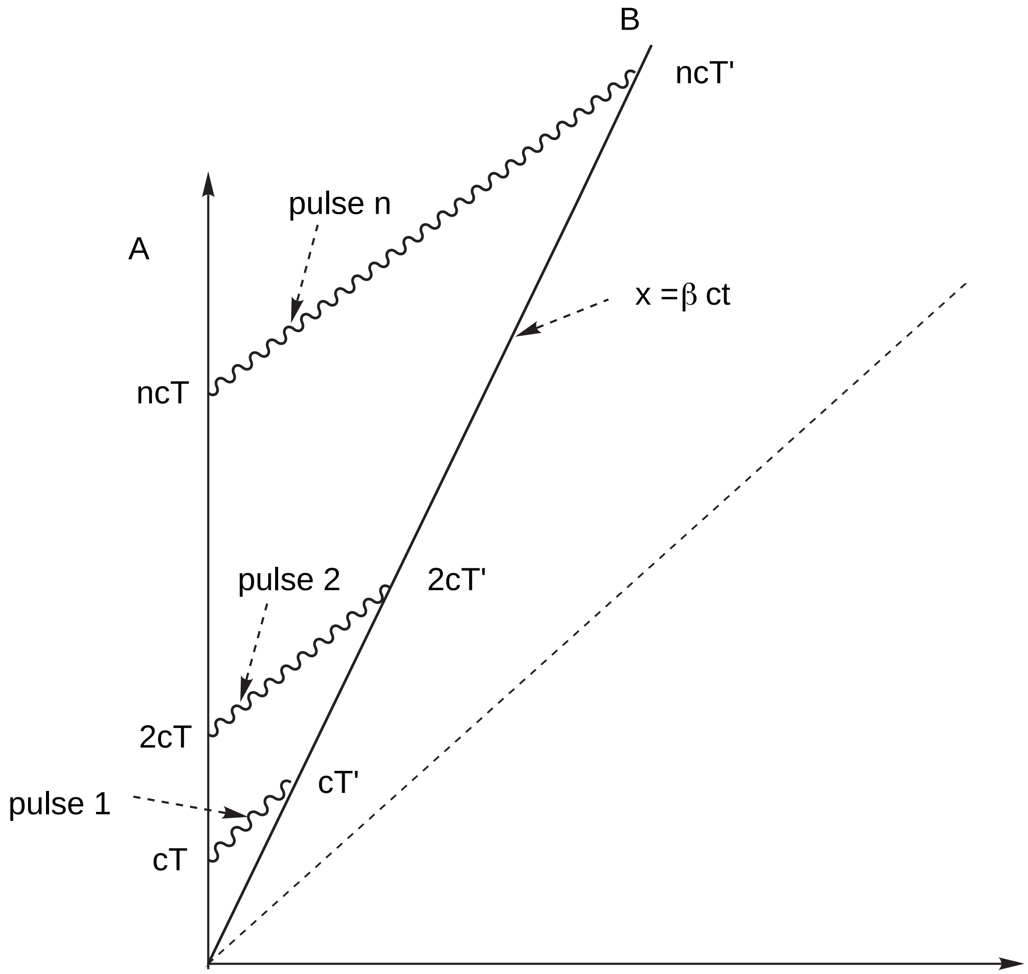
and

$$f_D = f_0\,\frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c}} = f_0\sqrt{\frac{c + v}{c - v}}$$

Here f_D is the frequency in the detector frame and v is the relative velocity of the source and the detector. **It does not matter which one is actually moving**!!

This result is just the red shift formula we started with earlier, as expected.

Now consider the spacetime diagram below. We have two observers in relative motion and the unprimed observers is sending signals to the primed observer at regular interval (separated by a time T).



The reception of the last pulse occurs at the point of intersection of the lines

$$x = c(t - nT)$$

$$x = \beta ct$$

or at

$$ct = \frac{cnT}{1-\beta} \quad , \quad x = \frac{\beta cnT}{1-\beta}$$

n pulses are sent out by the unprimed observer in nT seconds and thus the period is T seconds and the frequency is $1/T$.

n pulses(same number) are received by the primed observer in nT' seconds and thus the period is T' seconds and the frequency is $1/T'$.

Now, the reception point also corresponds to

$$ct' = \gamma(ct - \beta x) = \gamma\left(\frac{cnT}{1-\beta} - \frac{\beta^2 cnT}{1-\beta}\right)$$

$$= \gamma cnT \frac{1-\beta^2}{1-\beta} = ncT' = \gamma cnT(1+\beta)$$

Using

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

we get

$$T' = \sqrt{\frac{1+\beta}{1-\beta}}T = kT$$

which is the standard Doppler effect for light.

How Do We Talk to Each Other in this New Relativistic World?

In this new world what happens if we try to tell a story?

In particular, these are some of the words are no longer usable?

where, when, speed, distance, time interval,
simultaneous, same place, length, etc ...

If we want to use such words, then each reader (other observers) must first use the Lorentz transformations to translate the story before trying to read it!

The only words (concepts) that we are allowed to use if we do not want to do any translations are

interval, c, number of events

Not having grown up in this new world, we would find it very difficult to tell such a story.

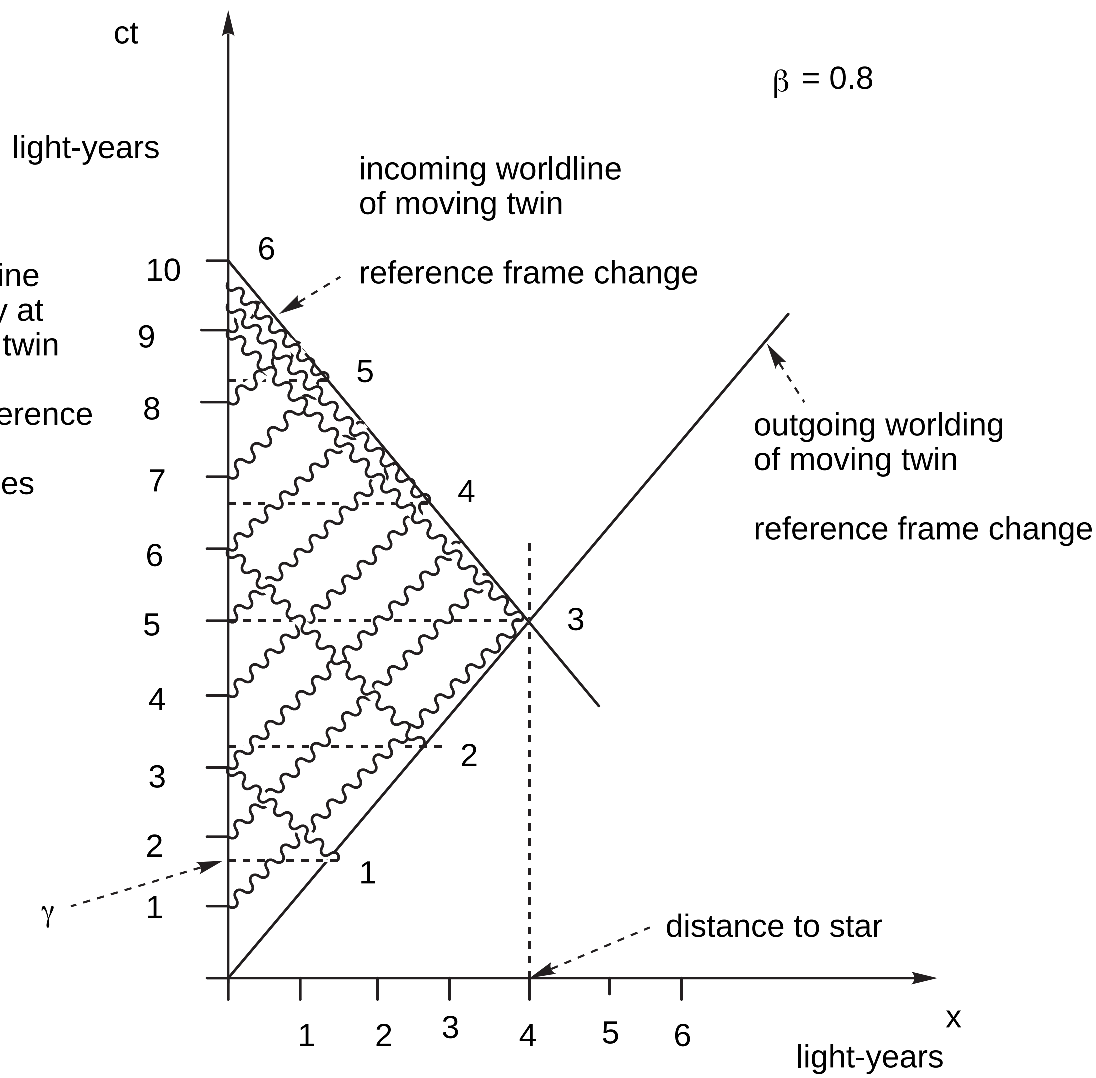
The Famous Paradoxes

The Twin Paradox - Let me state this problem in a **"bad"** way, i.e., a way that leads to the so-called paradox. Then we will state it correctly and the paradox will disappear and we will be able to draw the correct conclusions. **This might be a lesson for life also !**

Statement #1 - Two twins are traveling relative to each other with speed v. Time dilation says that the clock of the **"moving"** twin should tick slower (the time between ticks is larger). Since each twin considers herself to be at rest, the other twin should have a clock that runs slower and hence the other twin should be younger. Which twin is younger? There is no definite answer to the question as posed since we do not know which twin is moving (changed reference frames - has accelerated) and hence we have a supposed paradox.

Statement #2 - Two twins have been together since birth (they have been on the same worldline - in the same frame of reference). At one point in time, one of the twins, gets into a rocket ship and changes her frame of reference - changes her velocity - experiences a period of acceleration). The twin in the rocket ship travels to a distant star and then changes her frame of reference again (reverses her velocity - accelerates for a period of time). The twin in the rocket ship travels back to the earth and then changes her frame of reference again (come to rest on the earth - accelerates for a period of time). Finally, the two twins remain together again (in the same frame of reference - on the same worldline).

Which twin, if any, is younger? This description is represented by the spacetime diagram below:



On this diagram $\beta = 0.8$ and $\gamma = 1.67$. The respective time axes have been calibrated. Each twin sends one signal per year (by their own clock) to the other twin.

While they are separating, the k-factor says that

$$f_{observed} = f_{reduced} = \sqrt{\frac{1-\beta}{1+\beta}}1 = \frac{1}{3} \text{ per year}$$

While they are coming back together, the k-factor says that

$$f_{observed} = f_{increased} = \sqrt{\frac{1+\beta}{1-\beta}}1 = 3 \text{ per year}$$

It is clear from the diagram that both twins see these different rates during the designated periods.

For both twins the reduced rate starts immediately.

However, the switch over to the increased rate takes place at different times according to each observer. They are not identical observers and thus we should not expect identical results from their measurements.

For the moving twin, the switchover takes place exactly at the midpoint of the trip or at year 3 (as can be seen in the diagram). For the stay-at-home twin, however, the switchover take place at year 9.

Thus the moving twin sees 3 x 1/3 + 3 x 3 = 10 signals from the stay-at-home twin and thus knows that the stay at home twin is 10 years older and she is only 6 years older

The stay-at-home twin sees 9 x 1/3 + 1 x 3 = 6 signals from the moving twin and thus knows that the moving twin is 6 years older and that she is 10 years older

Both agree and this there is no paradox. The traveler ages less because moving clocks(clocks that have changed frames of reference) run slower.

Pole in the Barn

In this case we have the following situation - two farmers have a barn which is 10 meters long in their rest frame (unprimed). The farmers are standing at the left and right doors of the barn (the doors are open).

A pole carrier has a pole of length 12 meters in her rest frame and is carrying it horizontally while she runs towards the barn with a speed given by $\beta = 0.80$. This means that $\gamma = 1.67$.

If we believe all this relativity and length contraction stuff, then the farmers think the pole is

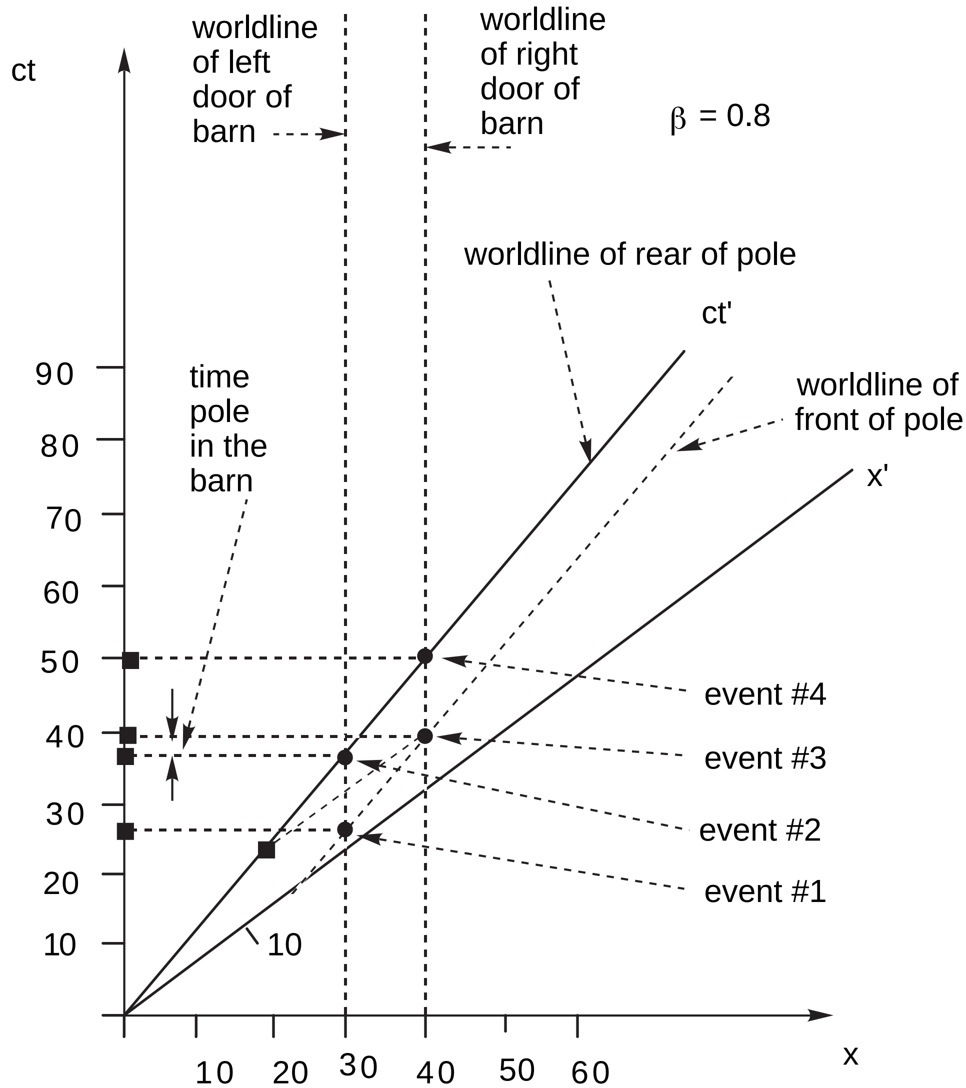
$$L_{pole} = \frac{L'_{pole}}{\gamma} = 9.8\,meters$$

However, the pole carrier thinks the barn is only

$$L'_{barn} = \frac{L_{barn}}{\gamma} = 8.0\,meters$$

This means that, according to the farmers, the pole should be able to fit into the barn. The pole carrier, however, say no way, the barn is much too small.

A possible spacetime diagram for this experiment is shown below.



Is there any correct answer to this dilemma? To answer the question, we label 4 crucial events:

event #1 = front of the pole enters the barn
event #2 = rear of the pole enters the barn
event #3 = front of the pole leaves the barn
event #4 = rear of the pole leaves the barn

These events are clearly shown on the diagram..

Now if $t_3 > t_2$ then the pole is completely within the barn for the period of time $t_3 - t_2$.

It is clear from the diagram, that according to the farmers the pole is within the barn for a short period of time!

The pole carrier disagrees, however. For the pole carrier, $t'_2 > t'_3$ and the pole is never completely within the barn.

There is a disagreement between the two sets of observers because the time order of the two crucial events (namely 2 and 3) has reversed.

Thus, both are correct.

The pole is within the barn and not within the barn depending on your frame of reference. Relativity is subjective, that is, dependent on the observer information in certain cases. Relativity allows different observers to tell differing stories like this when time order reverses. The time order reversal is OK in this case because events 2 and 3 are spacelike separated and thus reversing their time order cannot upset causality. There is no paradox!

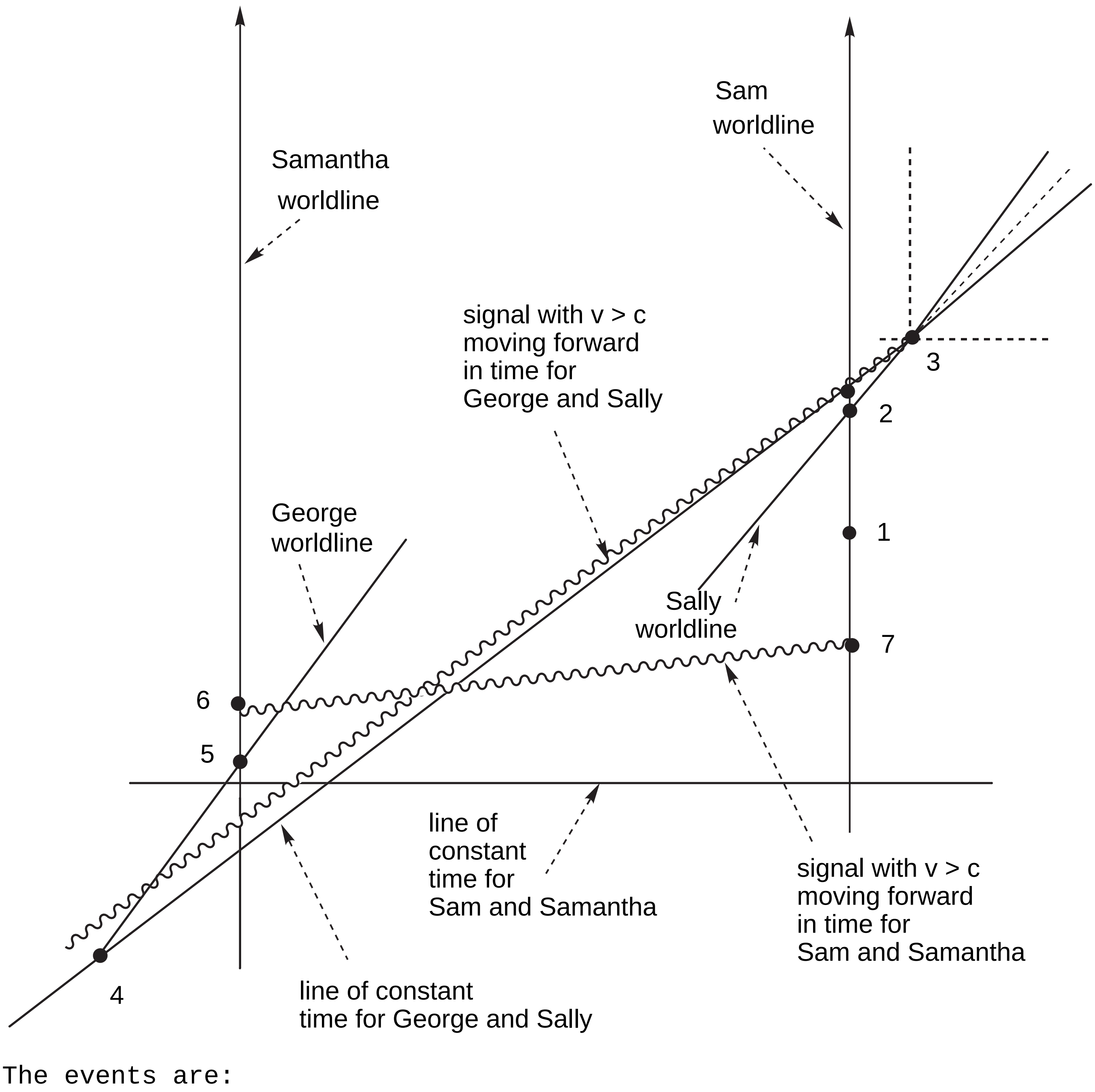
Faster than Light

What happens if we allow some signal to go faster than the speed of light?

Consider the following story.

Sam is walking down the path towards Sharples. As he passes near Clothier tower a stone block falls off the tower and lands on his head, killing him. So Sam is now lying in heap at the base of Clothier tower. Soon after that incident, Sally comes along. Sam is Sally's good friend and she is distraught when she sees Sam lying in a heap. Sally is walking past Sam with some speed u (she is in a different frame of reference). Now, Sally understands Special Relativity. Sally has in her possession a special device that can send a signal to someone on the other side of the universe at a speed $> c$ if they are in the same frame of reference. So Sally sends out a signal indicating what happened to Sam. The signal is received on the other side of the universe by George(in the same frame of reference as Sally). He is now desperate to tell Sam so he can avoid the stone block, but Sam is in a different frame and cannot receive his signal. So he tells the story to someone in Sam's frame, namely, Samantha. Samantha also happens to have one of those devices that sends the speedy signal and she sends a signal to Sam.

The entire sequence of worldlines with the associated events is shown in the diagram below:



The events are:

event #1 - Sam gets killed

event #2 - Sally sees Sam
event #3 - After patiently waiting Sally sends a $v > c$ signal to George
event #4 - George receives the signal
event #5 - George tells Samantha
event #6 - Samantha patiently waits and then send a $v > c$ signal to Sam
event #7 - Sam receives the signal from Samantha, realizes he is about to die and stops walking, thus avoiding the block and subsequent death

Questions:

If Sam is not dead, why would Sally send any signal?

If Sally does not send a signal making all the other stuff happen, then why would Sam stop?

If Sam has no reason to stop, he then gets killed and Sally has a reason to send the signal.

Which is it?

We have what is called a **closed causal loop** here. There is no logical way out of this loop.

Does that mean it cannot occur, i.e., that no signal can travel faster than light?

or

Is there some other explanation?

General Relativity

In special relativity we found that the spacetime interval or just "interval" between two events

$$\#1 : (x, y, z, ct)$$
$$\#2 : (x + \Delta x, y + \Delta y, z + \Delta z, c(t + \Delta t))$$

is given by

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

We can generalize this result to

$$\Delta s^2 = \sum_{i=0}^{3} \sum_{j=0}^{3} g_{ij} \Delta x_i \Delta x_j$$

where

$$x_0 = t\,,\; x_1 = x\,,\; x_2 = y\,,\; x_3 = z$$
$$g_{00} = 1 = -g_{11} = -g_{22} = -g_{33}$$
$$g_{ij} = 0 \quad if \quad i \neq j$$

In the case of special relativity, this is just a change in notation and all the g_{ij} (called the **metric components**) are constants.

The equation for the light cone for the event (x,y,z,ct) can be expressed

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = 0$$

This says that spacetime itself is **"flat".** This means that the shortest distance between two points in space is a straight line.

Now what happens if the metric components g_{ij} are not constants but are functions of space and time?

First, the Lorentz transformations are no longer valid.

Second, the "light cone" will look different at different points in spacetime.

Third, spacetime itself is **"curved".** This means that the shortest distance between two points in space is not a straight line.

Einstein in his theory of gravitation (called General Relativity) proposed that

$$F(g_{ij}) = G(Energy, Momentum)$$

i,e., that the metric components g_{ij}, which determine the shape of the light cones in spacetime, are determined by the distribution of energy and momentum in spacetime

or

the very structure of spacetime is determined by the energy density in spacetime.

A result of the theory is that light would be affected by gravitational fields. The following predictions were made and have been confirmed experimentally :

(1) If we place a light source at the top of a tower and shine the light downwards, then the change in the strength of the gravitational field as we go from the top to the bottom of the tower causes a gravitational redshift such that

$$\frac{f_{h+\Delta h} - f_h}{f_h} \approx \frac{GM}{R_{earth}^2}\Delta h \approx 10^{-14} \quad \text{for a 10 m tower}$$

(2) If we place a clock at the top of a tower and a clock at the bottom, then because of the difference in the strength of the gravitational field between the top and the bottom of the tower the clocks run at different rates - called the gravitational time dilation

$$\frac{\tau_{h+\Delta h} - \tau_h}{\tau_h} \approx \frac{GM}{R_{earth}^2}\Delta h$$

(3) If light passes by a large mass (like a star) it does not travel in a straight line but is bent. The amount of bending has two observable consequences

(a) if a star is observed when the sun is not in the way and then when the light would just pass by the sun, the observed difference in direction to the star is about 1.75 seconds of arc

(b) if a signal is sent from Earth to Venus with the sun in between, there is a time delay due to longer(bent) paths of motion of about 1.1×10^{-4} sec

(4) Galaxies can cause gravitational lensing which results in double images for distant stars

(5) The long axis of the planetary orbit ellipse in the solar system precesses - for mercury this is about 43 seconds of arc per century

All particles in this theory are free particles, i.e., there are no forces.

All particles move along geodesics, which are the path of shortest interval in spacetime.

The geodesics for a given spacetime are determined by the metric components. So the distribution of energy determines the metric components which in turn determines the geodesics and particles move on geodesics.

In flat spacetime (think of a plane in space) the geodesic is a "straight" line. In fact, the geodesic is always the "straightest" line in a given spacetime. In curved spacetime (think of the surface of a sphere) the geodesic is not a straight line(great circle on the sphere).

If we move a vector "parallel" to itself over a closed curve in flat spacetime it does not change its direction.

If we move a vector "parallel" to itself over a closed curve in curved spacetime it does change its direction.

If I turn off gravity and throw an eraser, then it follows the geodesic in this "flat" spacetime which is a straight line.

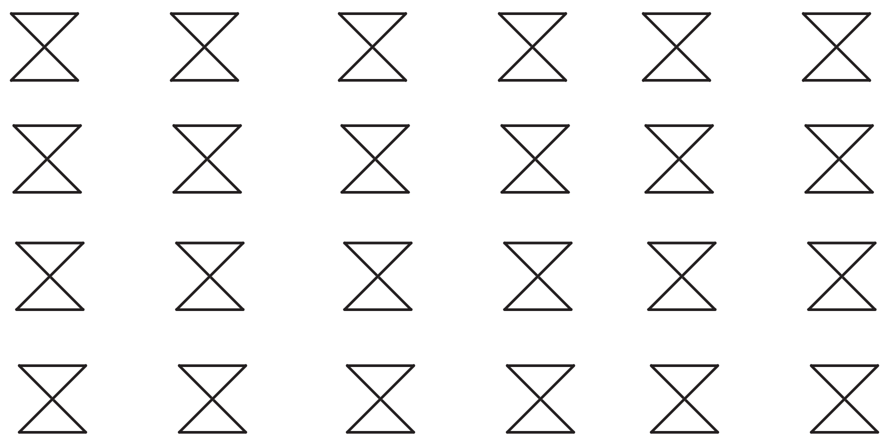
If gravity is present, then it follows the geodesic in this "curved" spacetime which is a parabola.

The planetary elliptical orbits in space are the geodesics for the 4-dimensional spacetime near the sun.

All of these result have been **confirmed experimentally**.

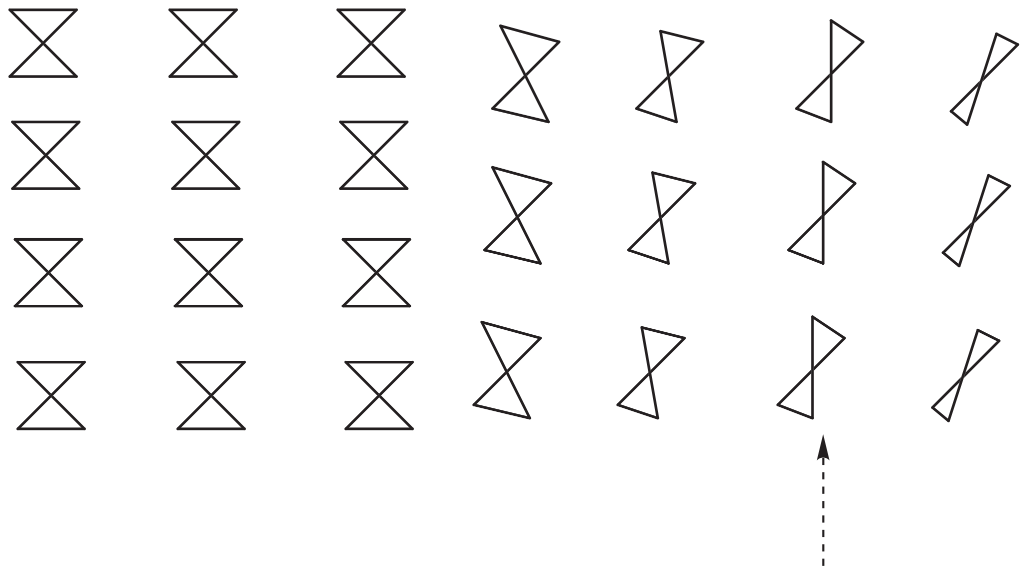
Black Holes

In special relativity the light cone structure of spacetime looks like



This what we mean by **"flat space"** ....the light cones are the same everywhere.

What happens, however, if we observe a light cone structure of spacetime that looks like



Then, to the left we have flat spacetime, but to the right something strange is happening. The left side of the light cone is rotating clockwise.

This means that access to regions to the left is being restricted (takes longer to get there).

As we go further to the right, we reach a point (arrow) where the left side of the cone is vertical and all of space to the left is no longer accessible. This point is on a surface called an **"event horizon"**.

Once some observer crosses this surface we can no longer see them(there is an infinite redshift) and they (and light) can no longer get to the back across the surface (hence the name "black" hole).

The observer can only proceed(remember must stay inside forward cone) to the right where the light cones tilts even further.

The end result is the light cone being a single line and the observer having no choice about future motion. This point is called a "singularity".

Long before reaching the singularity, the tidal forces become so large that any object is torn apart.

These radical solutions to Einstein's equation have now been confirmed experimentally (via the radiation coming from matter falling into the black hole) and are thought to exist at the center of all galaxies.

In a static black hole as just describe, the event horizon and the infinite red shift surface are the same surface and energy can only pass through in one direction.

If the black hole is rotating, however, the event horizon and the infinite red shift surface are not necessarily the same surface. The regions between an infinite red shift surface and a event horizon is called the **ergosphere**.

It is possible to extract energy from the ergosphere as follows:

(1) a spaceship falls from infinity into ergosphere along an orbit with positive energy
(2) once there, using a spring-loaded device we eject a brick into an orbit with negative energy
(3) the spaceship recoils into a new orbit with larger positive energy
(4) energy is constant(conserved) so the spaceship emerges with more energy then it went in, but the black hole + brick have lost energy

A very tricky and dangerous maneuver.

Now it is possible to follow a worldline that is everywhere timelike(allowed) such that one passes through the ergosphere and the particle emerges before it entered ($\Delta t < 0$).

The time change can be made arbitrarily large by completing orbits inside the ergosphere ..... this is a model of a time machine for travel to the past!!

This violates causality, however, and results in a logical contradiction.

Consider the particle to be a signal (a signal rocket) that is emitted at $t = 0$ by an apparatus located far from the rotating black hole (where spacetime is flat), but is received by this same apparatus at an earlier time, say $t = -2$.

Suppose the apparatus is programmed with the following instructions:

(1) emit a signal if the signal is not received before $t = 0$
(2) do not emit a signal if the signal is received before $t = 0$

This implies a logical contradiction with emission at $t = 0$ and reception at $t = -2$!!

So something will have to give!!!! Very fundamental and exciting stuff......

Basic Ideas of Classical Kinematics and Dynamics (A Quick Tour)

Kinematics (or the study of motion in time)

Position $(\vec{r}(t))$ is defined as a vector from the coordinate origin to the 3-dimensional point where the object is located. In 1-dimension we have x(t) .

The goal of all classical physics is to determine the **position** of an object as **function of time**.

Position answers the **"Where"** question for the events we have been discussing.

Velocity $(\vec{v}(t))$ is defined as a vector in the direction of the change of the position vector and having a magnitude given by

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} \quad \text{or} \quad v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \quad \text{in 1-dimension}$$

The direction of the velocity is always **tangent** to the path of motion.

Velocity tells us how fast the object is moving and in what direction.

If the velocity is constant this means both its magnitude (speed) and direction are constant because it is vector. This is the easy case, for example,

Suppose $v = +10\, m/s$ towards $+\infty$ and $x = 2\, m$, where will particle be 1 sec later.

Clearly the answer is $x = 12\, m$ since 12 = 2 + 10x1 = $x(0) + v\Delta t$.

If the velocity is not constant the situation is more complicated (Physics 7). If, however, I can tell you that the **average velocity** over the next second = $8\, m/s$, then the rule

12 = 2 + 10x1 = $x(0) + v\Delta t$

still works.

So, in general, we have

$$x(t + \Delta t) = x(t) + v(t)\Delta t \quad \text{in 1-dimension}$$

where $v(t)$ = the average velocity in the interval $t \to t + \Delta t$.

In 1-dimension direction is indicated by ± signs.