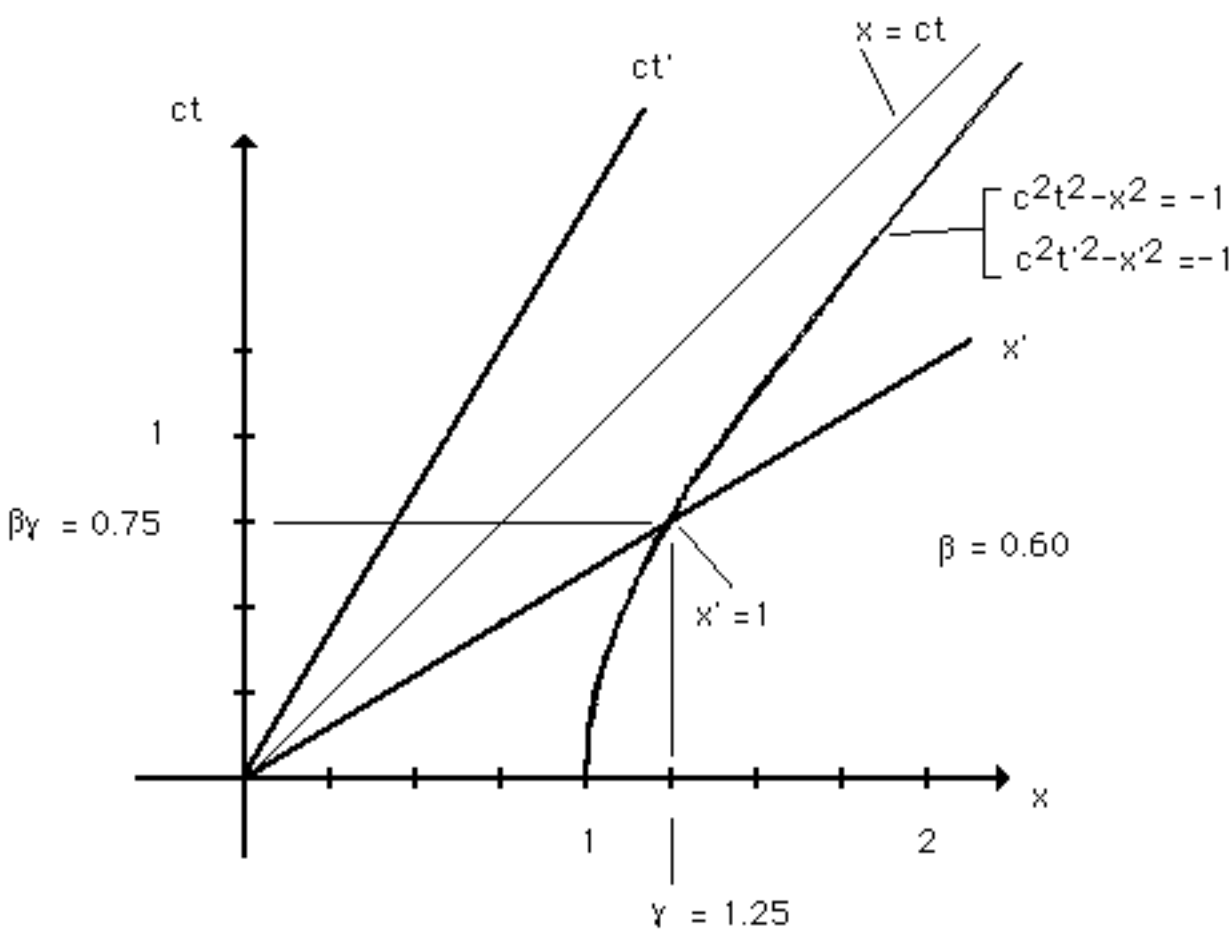
where

$$x' = \gamma(x - \beta ct)$$
$$ct' = \gamma(ct - \beta x)$$

and we have used the invariance of the spacetime interval. The set of all events that satisfy these equations is a curve on the spacetime diagram

This curve is a hyperbola (see the diagram below).

It intersects the x-axis at $x = 1$(when $ct = 0$) and the x'-axis at $x' = 1$ (when $ct' = 0$) and allows us to calibrate the x'-axis once we have calibrated the x-axis (or vice versa). For diagram construction convenience we note that the point $(ct = \beta\gamma, x = \gamma)$ corresponds to the intersection determining the point $x' = 1$ as shown on the diagram.



In a similar manner, the ct'-axis is calibrated in terms of the ct-axis using the curves

$$(\Delta S)^2 = c^2(\Delta t)^2 - (\Delta x)^2 = c^2t^2 - x^2 = +1$$

$$(\Delta S')^2 = c^2(\Delta t')^2 - (\Delta x')^2 = c^2 t'^2 - x'^2 = +1$$

It intersects the ct-axis at $ct = 1$(when $x = 0$) and the ct'-axis at $ct' = 1$ (when $x' = 0$) and allows us to calibrate the ct'-axis once we have calibrated the ct-axis (or vice versa). For diagram construction convenience we note that the point $(ct = \gamma, x = \beta\gamma)$ corresponds to the intersection determining the point $ct' = 1$.

Note that light rays are $45°$ lines on the Minkowski diagram(because of our scale choice).

Alternatively, we can use an experimental result to calibrate the time axis and then assume by symmetry that the space axis calibrates in the same manner. This experiment involves the decay of an elementary particle called a mu-meson.

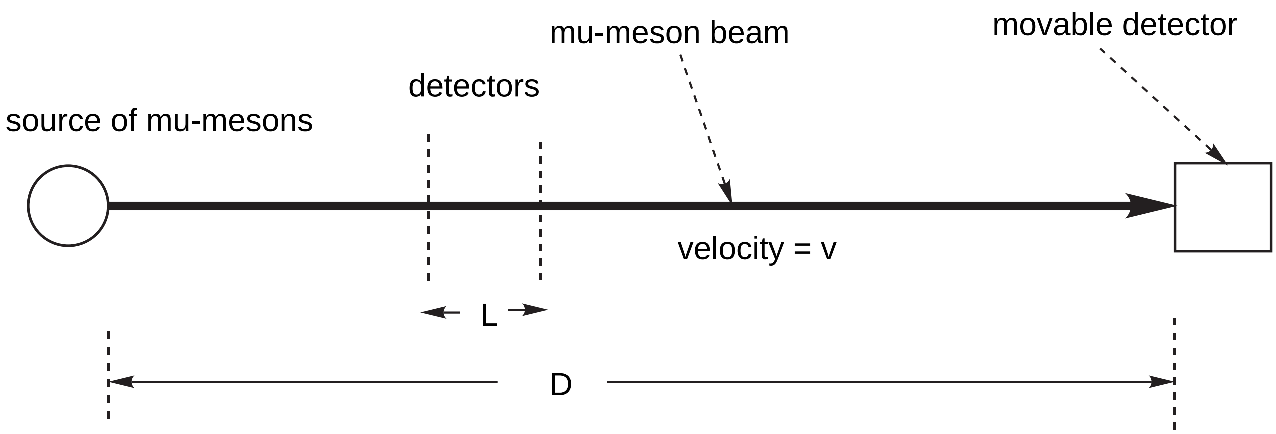
A mu-meson is a short-lived elementary particle that is produced in large numbers at the top of the atmosphere when the atmosphere is struck by a high-energy cosmic ray particle. Mu-mesons can also be produced in large numbers at any accelerator laboratory.

Experimentally, if the mu-mesons are produced in the laboratory at rest ($v = 0$), then they only live for a very short time of about $\tau_0 = 2x10^{-6}\sec$ = 2 microseconds = 2 ms = their lifetime at rest. Since we have already decided that no object can have a speed greater than c = $3x10^8 m/\sec$, the maximum distance the mu-mesons can travel during their lifetime before they decay into an electron and a neutrino is about $c\tau_0 = 600m$. In this calculation, we have explicitly assumed absolute time, which says that the lifetime of a moving mu-meson is the same as that of a mu-meson at rest (we now know this is not true).

The first experimental indication that absolute time was a false concept came from these mu-mesons produced by cosmic rays. Since they are produced at the top of the atmosphere and can only live to travel a maximum of 600 m and since the atmosphere is about 10,000 m thick, no mu-mesons should be observed on the ground (certainly only a small number compared to the number at the top of the atmosphere).

Experimentally, however, the number at the top is the **same** as the number at the bottom. So something is extending the lifetime of the mu-mesons.

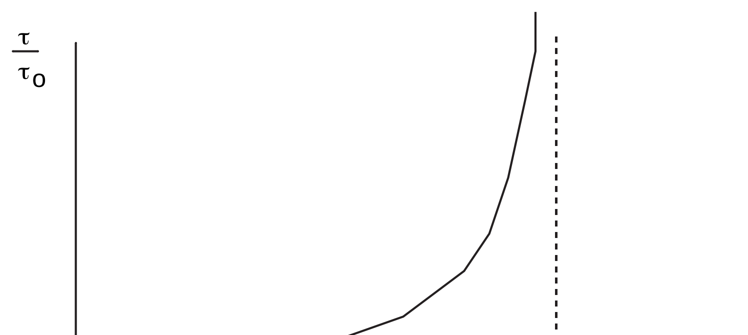
In the laboratory we can do this experiment with precision. The setup is as shown below:



A beam of mu-mesons a sent from the source to a movable detector a distance D away. Along the way two detectors a distance L apart measure the time Δt it takes the mu-mesons to travel the distance L. This determines their velocity

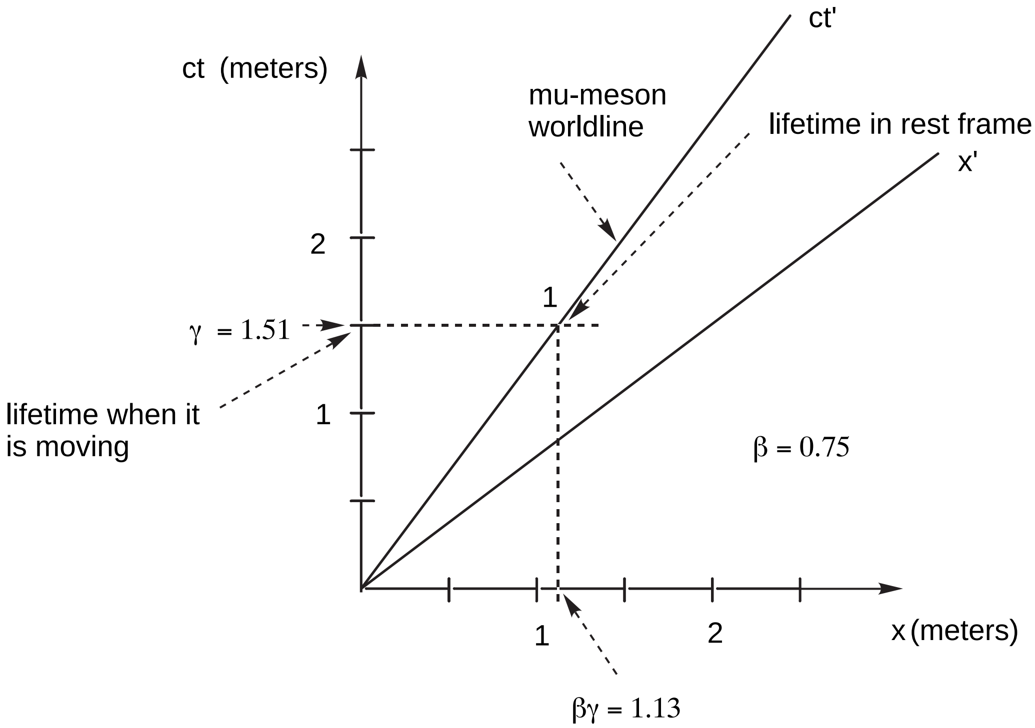
$$v = \frac{L}{\Delta t}$$

If absolute time were correct, then after a distance $d = v\tau_0$ all the mu-mesons should decay and none should be seen in the movable detector if $D > d$. The experimental result is that the mu-mesons travel a maximum distance = $v\tau$ where τ is the lifetime of the moving mu-meson. The experiments found that $\tau = \gamma\tau_0 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\tau_0$. A plot of this result looks like:

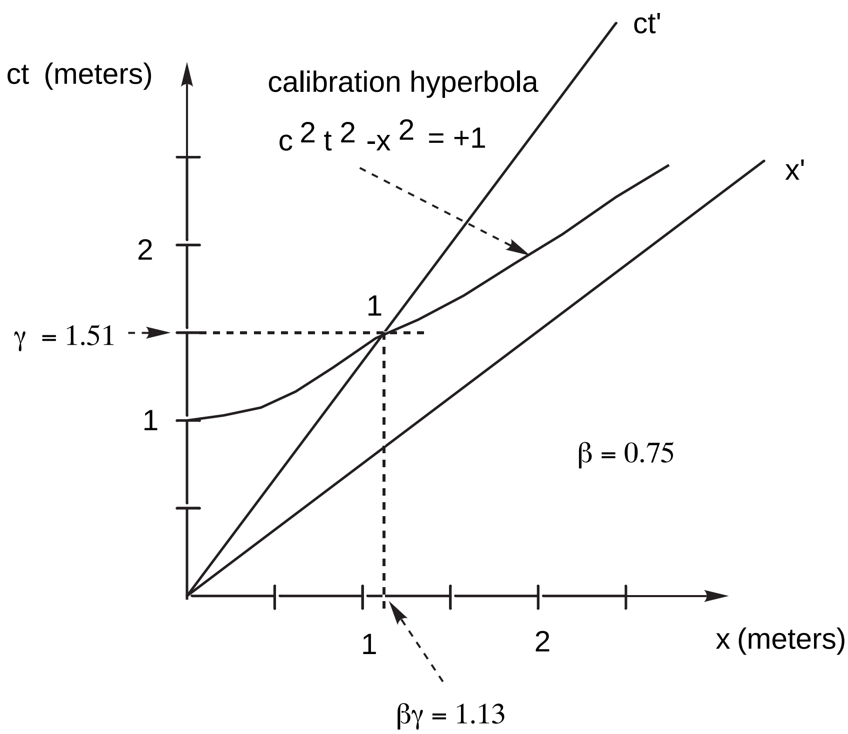


The lifetime gets larger and larger the closer the velocity approaches the speed of light!!!!

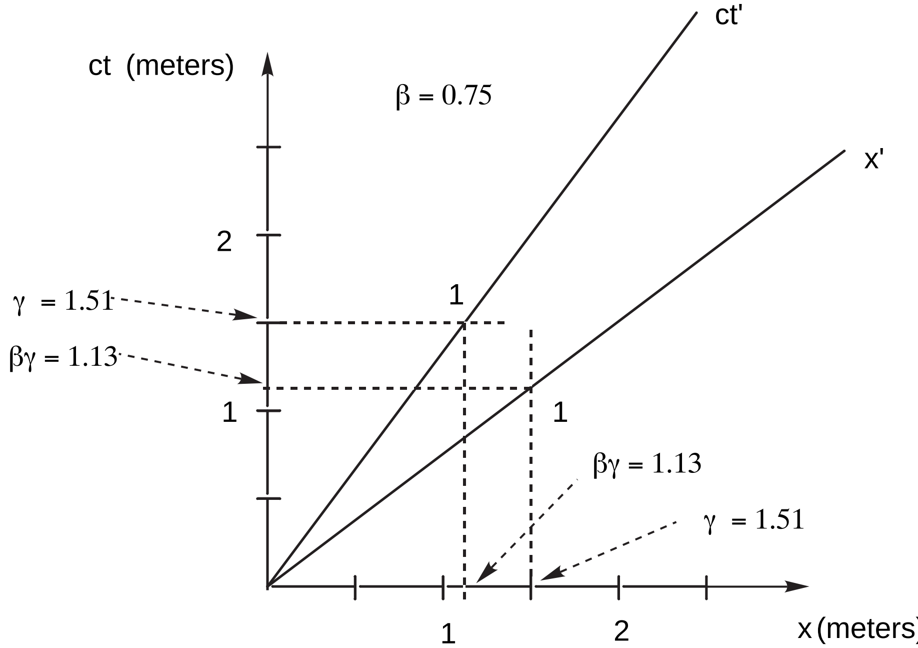
If we let τ_0 = 1 tick of a clock(the clock vanishes after a single tick!) and let the mu-meson travel with the primed observer, then these experimental results are represented by the diagram below:



So we see that the calibration procedure using the invariance of the spacetime interval agrees with this experimental result.



General Spacetime Diagram Construction Procedure



As shown in the diagram above we carry out these steps:

(1) Set up orthogonal(perpendicular) x- and ct-axes .
(2) Choose identical scales for these axes (units(meters, light-seconds, light-years, etc) are chosen appropriate to the problem at hand)
(3) Locate the point ($\beta\gamma,\gamma$) on these axes.
(4) Draw a line from the origin $(0,0)$ through this point. That is the x'-axis.

The point ($\beta\gamma,\gamma$) is the point ($x'=1, ct'=0$) so this calibrates the axis.

(5) Locate the point ($\gamma,\beta\gamma$) on these axes.
(6) Draw a line from the origin $(0,0)$ through this point. That is the ct'-axis.

The point ($\gamma,\beta\gamma$) is the point ($x'=0, ct'=1$) so this calibrates the axis.

The diagram is just a visual representation of the Lorentz transformation equations. It is a view of all spacetime (past, present and future).

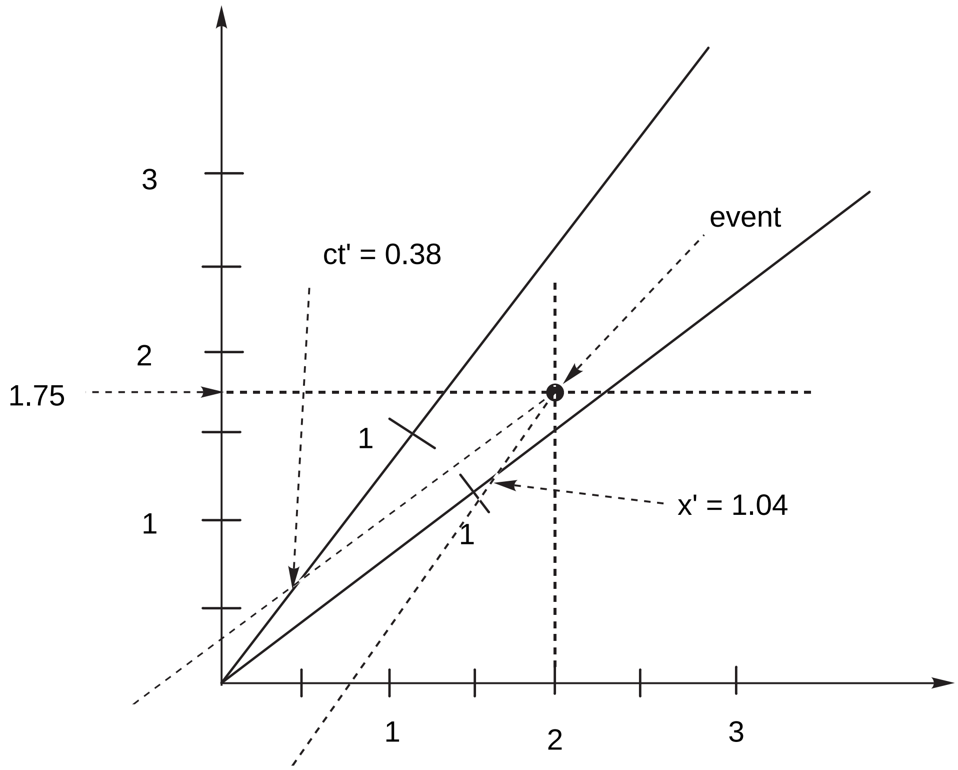
To see that it it agrees with the Lorentz transformations let us do an example.

Suppose that $\beta = 0.75$. Then we have $\gamma = 1.51$ and $\beta\gamma = 1.13$. The spacetime diagram looks like the figure above. Now consider an event

$(x = 2.0, ct = 1.75)$. The Lorentz transformations say that the other observer sees the event

$$x' = \gamma(x - \beta ct) = 1.51(2.0 - 0.75(1.75)) = 1.04$$
$$ct' = \gamma(ct - \beta xt) = 1.51(1.75 - 0.75(2.0)) = 0.38$$

This result is confirmed by the diagram below:



Note that in order to find the primed coordinate values we must draw lines **parallel** to the primed-axes.

We now have a theory called **Special Relativity**. We can represent it either by the Lorentz transformations, the invariance of the interval or the Minkowski spacetime diagram. All representations of the theory are equivalent. It was discovered by Albert Einstein in 1905.

What is a theory? A theory is a set of assumptions that

(1) agree with a set of known experiments (three in our case)
(2) lead to predictions (correct) for all **new** experiments

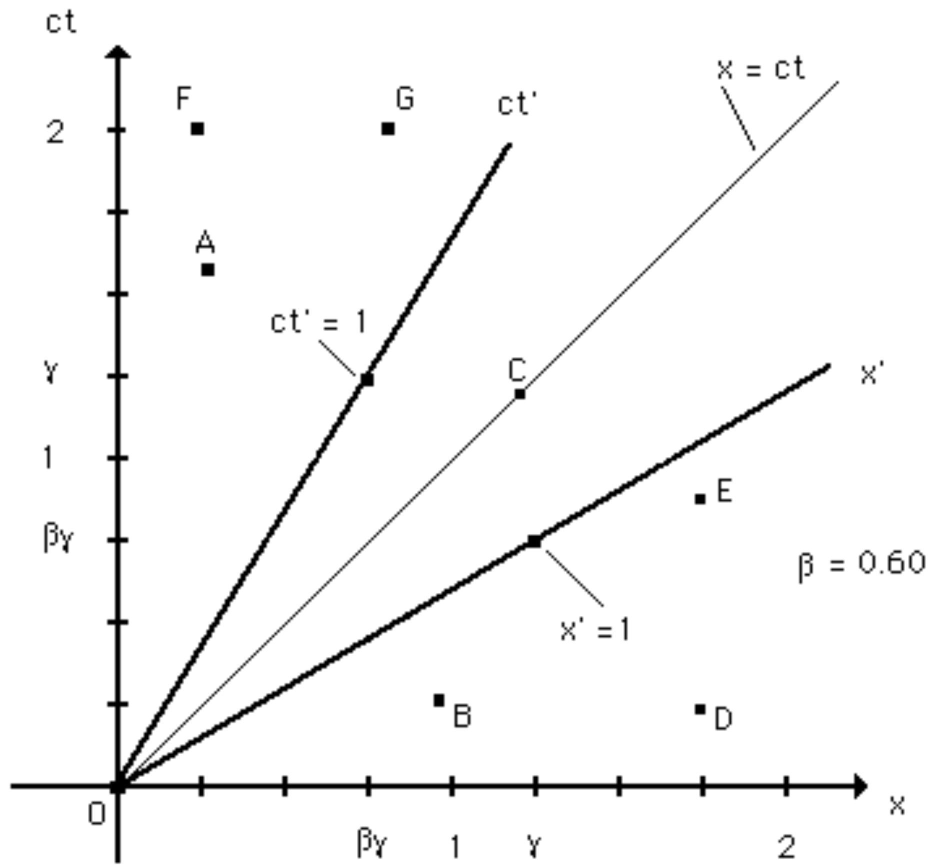
Newton-Galileo physics lasted over 250 years before any experiment was sophisticated enough to show that it was invalid. Special Relativity has now lasted 100 years.

It has been subjected to significantly more experiments than was case for the Newton-Galileo theory. These experiments are significantly more sophisticated and more precise also.

What are the new predictions of the theory?

The Strange World of Special Relativity

Consider the events labeled O, A, B, C, D, E, F, and G on the spacetime diagram below:

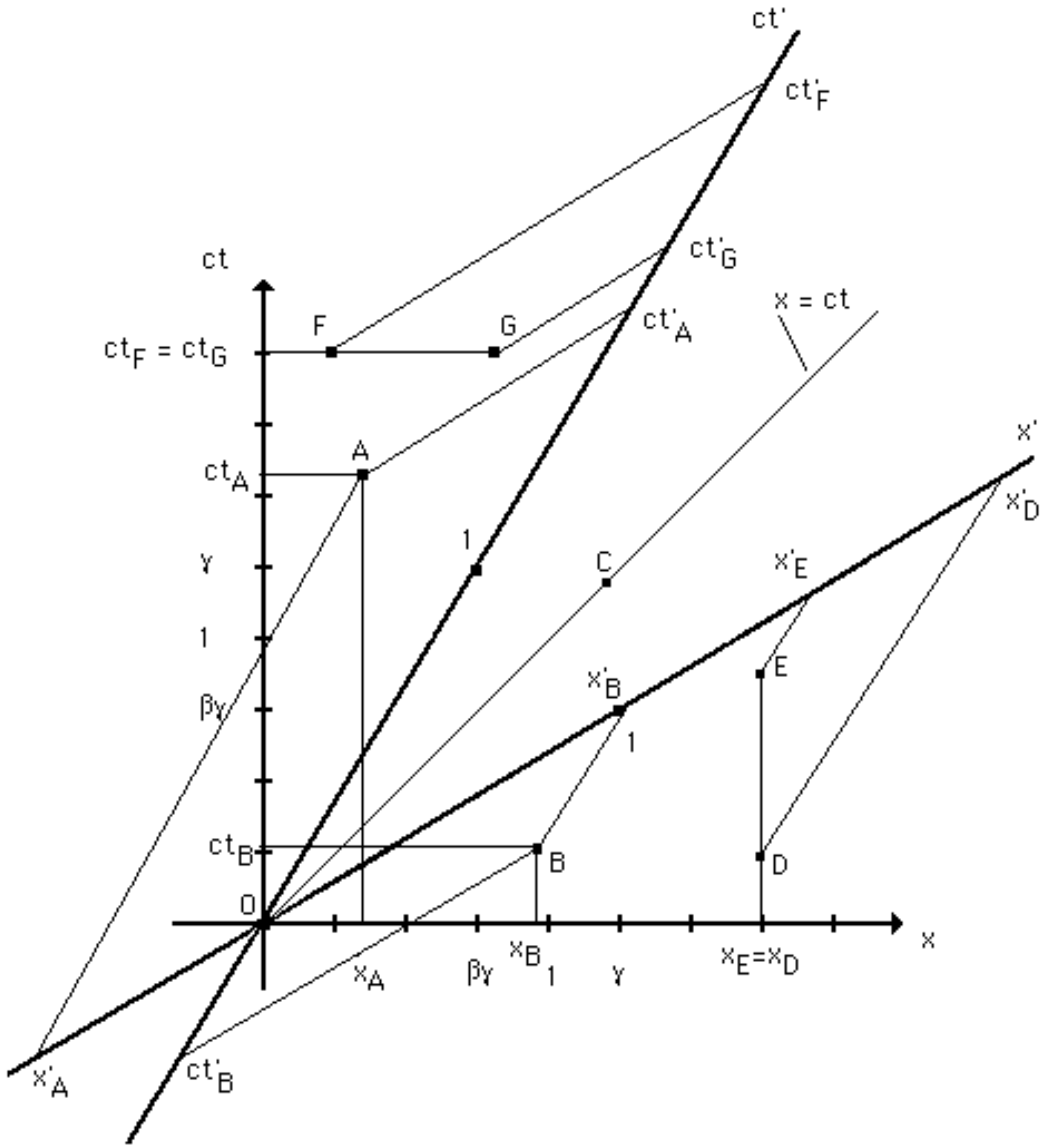


The corresponding intervals have the following properties:

$$(\Delta S)^2_{AO} = c^2(t_A - t_O)^2 - (x_A - x_O)^2 > 0 \rightarrow \text{a "timelike" interval}$$
$$(\Delta S)^2_{DO} = c^2(t_D - t_O)^2 - (x_D - x_O)^2 < 0 \rightarrow \text{a "spacelike" interval}$$
$$(\Delta S)^2_{CO} = c^2(t_C - t_O)^2 - (x_C - x_O)^2 = 0 \rightarrow \text{a "lightlike" or "null" interval}$$

$$(\Delta S)^2_{FG} = c^2(t_F - t_G)^2 - (x_F - x_G)^2 = -(x_F - x_G)^2 \rightarrow \text{ F and G are simultaneous in (x,ct) frame}$$
$$(\Delta S)^2_{ED} = c^2(t_E - t_D)^2 - (x_E - x_D)^2 = c^2(t_E - t_D)^2 \rightarrow \text{ F and G are at the same place in (x,ct) frame}$$

Now we consider these same events from the viewpoint of the (x',ct') frame. Look at the diagram below where we have marked off all of the coordinate values:



Looking carefully at this diagram we can draw the following conclusions:

[1] events that are simultaneous in one frame are not simultaneous in other frames (see events F & G) -

simultaneity is a relative concept!

[2] events occurring at the same place in one frame do not occur at the same place in other frames (see events E & D)

[3] the time order of timelike events (events with a timelike interval) does not change between frames (see events 0 & A)

[4] the time order of spacelike events (events with a spacelike interval) **can have their time order reversed** (see events O & B); in the (x,ct) frame B occurs after O, but in the (x',ct') frame O occurs after B.

[5] numerical values of spatial separations and time separations are different in different frames

[6] note that the line $x = ct$, which represents a light ray starting at the origin in the unprimed frame is also the line $x' = ct'$, which represents a light ray starting at the origin in the primed frame.

Light is the only physical object that both observers see in identical fashion.

Let us consider in more detail the **reversal in time order** of two events.

This seems to be a very serious problems since it could possibly lead to a violation of the idea of **causality.** The concept of **causality** is connected with the idea of **cause and effect**, i.e., that an event should not occur **before** its own cause, for example, a firecracker should not explode **before** we light the fuse!

Suppose that we have two events in the (x,ct) frame with coordinates (x_1,ct_1) and (x_2,ct_2) and suppose, in addition, that

$$\Delta x = (x_2 - x_1) > 0 \quad , \quad \Delta t = (t_2 - t_1) > 0$$

so that event 2 comes after event 1 in the unprimed frame. Then the Lorentz transformations give the result (in the (x',ct') frame) that

$$\Delta t' = \frac{1}{c}\left(ct'_2 - ct'_1\right) = \frac{1}{c}\left(\gamma(ct_2 - \beta x_2) - \gamma(ct_1 - \beta x_1)\right) = \gamma((t_2 - t_1) - \frac{\beta}{c}(x_2 - x_1))$$
$$= \gamma(\Delta t - \frac{\beta}{c}\Delta x)$$

It is easy to see that $\Delta t'$ can be negative, which means that the time order of the two events is reversed, if the two events are related such that we have

$$\Delta t - \frac{\beta}{c}\Delta x < 0 \quad \text{or} \quad \frac{\Delta x}{\Delta t} > \frac{c}{\beta} > c$$

or the events must be connected by a signal with $v > c$, which means that they are spacelike separated!

Now, for all timelike related pairs of events we have

$$\frac{\Delta x}{\Delta t} < c$$

and thus we **cannot reverse their time order**.

It is important to note that it is only for timelike related events that can event #1 cause event #2 (since all signals must have $v < c$). Thus, all cause/effect related events cannot have their time order reversed, preserving the idea of causality.

Special relativity is consistent with causality without us having to impose the consistency!

On the other hand, all spacelike related pairs of events have

$$\frac{\Delta x}{\Delta t} > c$$

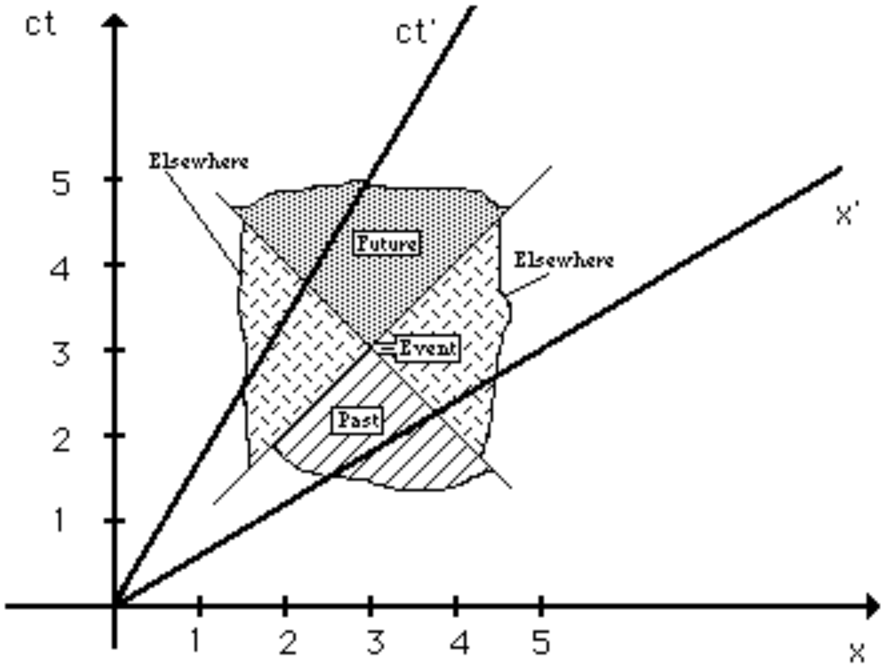
and thus, their time order might be reversed in different frames.

Since they cannot be cause/effect related, this does not affect the idea of causality. It does, however, lead to a number of strange **"paradoxes"**, as we shall discuss later.

Another way to look at these ideas is via the concept of the **light cone**.

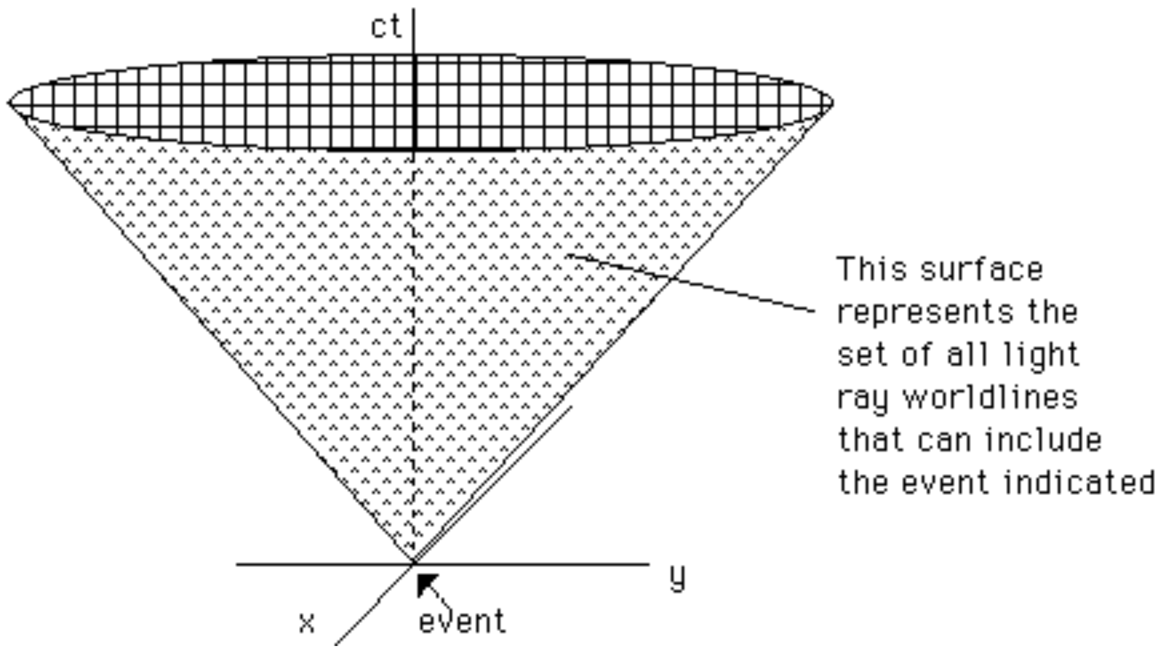
Since the maximum allowed speed for any physical object is the speed of light c, we can use the world lines of light emanating from an event to delineate distinct regions of spacetime for any object having that event on its worldline.

Consider the diagram below:



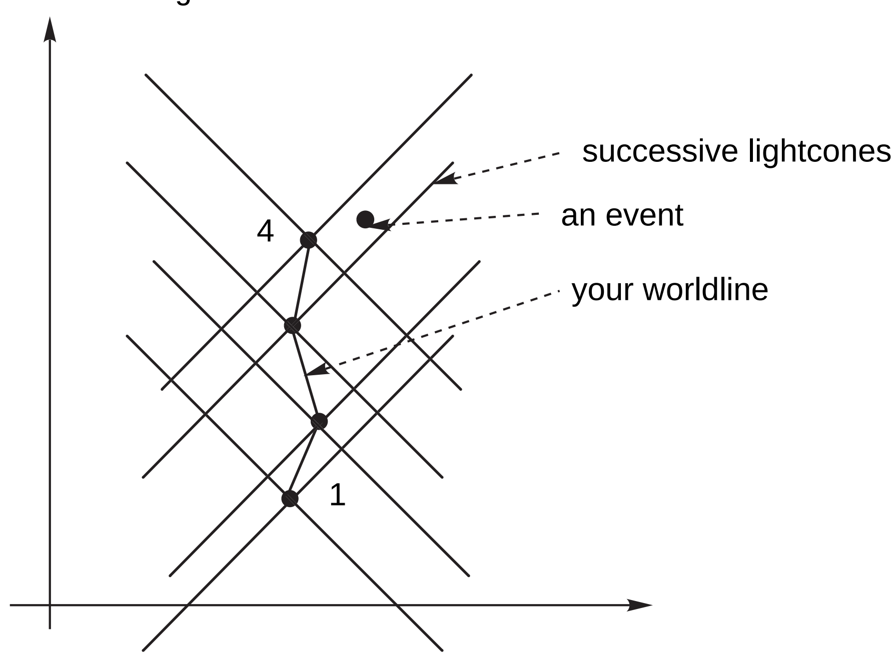
Suppose that you experience the event (that is, it is on your worldline) as indicated on the diagram above. Since neither you nor any signals you send or receive can travel faster than the speed of light and since the light ray worldlines containing this event are $45°$ lines as shown, the region labeled **"future"** represents all the events that you can either experience or influence with a signal at a later time(all events in this region are timelike separated from the event you experienced), the region labeled **"past"** represents all the events you could have experienced or that could have influenced you(all events in this region are timelike separated from the event you experienced). The regions labeled **"elsewhere"** are such that you can neither experience them nor influence them with any signal (all events in these regions are spacelike separated from the event you experienced).

If we draw this picture in a 3-dimensional world (x, y, ct) then the corresponding regions would look like:



and hence the name **"light cone"**.

What has happened to your possible future while we have been discussing these light cones?



The event labelled above was in your possible future when you were experiencing event #1, but is no longer in your possible future when you are experiencing event #4. So be careful about wasting time doing nothing!!