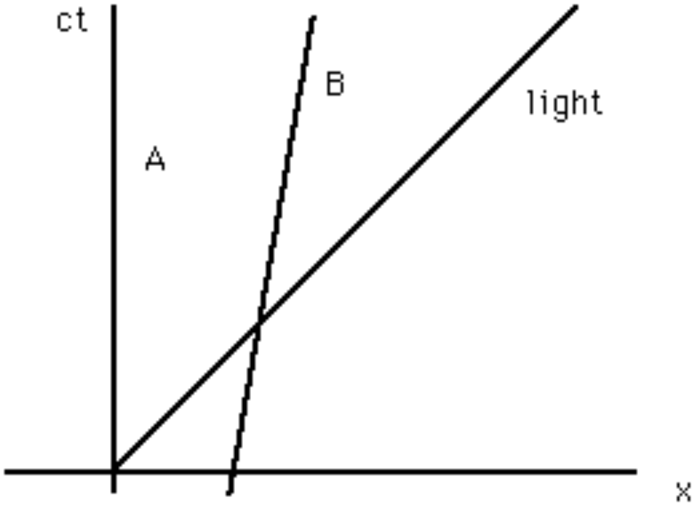
SpaceTime Diagrams

We consider two observers A and B. Observer B is moving **away** from observer A with constant speed v (we consider only 1-dimensional motion for simplicity). This is represented by the spacetime diagram shown below:



We have also included the worldline of a light beam that started at $(0,0)$.

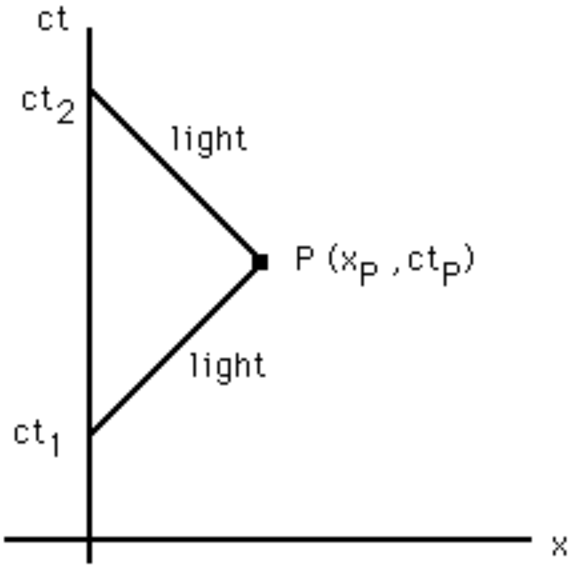
It is now clear why we choose the vertical axis to be ct rather than just t - the world line of light is then always a $45°$ line! We assume that each observer carries their own clock. In Moore, the same result is obtained by choosing $c = 1$.

As we shall see from the results of our derivation, each observer will need to determine the events on the worldline of the other observer using **only** measurements available **on their own worldline**. We will not be able to trust any information that is not recorded by instruments moving with us (on the same worldline). This means that we must figure out how A (or B) can measure the (x,ct) values for an event not on their own worldline.

Remember, what is true on my own worldline,namely, that the time interval between events that I experience(my worldline passes through them) is directly measured by the clock that I carry and my position is constant (usually assumed to be zero).

The method we now develop is called the "radar" method.

Now consider the diagram below:



Observer A assigns coordinates to the event P by bouncing a light signal off of whatever is occurring at P. The light signal is sent out at the event $(0, ct_1)$ and received back at the event $(0, ct_2)$. Note the important fact that both of these events are on A's worldline. We then have (using $\Delta x = c\Delta t$ for light)

$$(x_P - 0) = c(t_P - t_1) = c(t_2 - t_P)$$

or

$$ct_P = \frac{c(t_2 + t_1)}{2}$$

which is the **average** of sending and receiving times(makes sense). Then, substituting, we obtain

$$x_P = \frac{c(t_2 - t_1)}{2}$$

Thus, any observer (a particular worldline) can determine the coordinates of an event off that worldline by **only using light**, which has constant speed, by assumption, for all observers and only measuring time values on their own clock (clock is on same worldline). I emphasize again that this is crucial..... we must only use information about events we actually experience (that are on our worldline), otherwise we cannot be certain of their validity.

Special Relativity

We now use this procedure and our assumptions to derive a new theory called Special Relativity (this was done by Einstein in 1905).

Consider the experiments represented by the worldlines in the spacetime diagrams below. In each case, observers A and B are assumed to be moving away from each other with speed v.

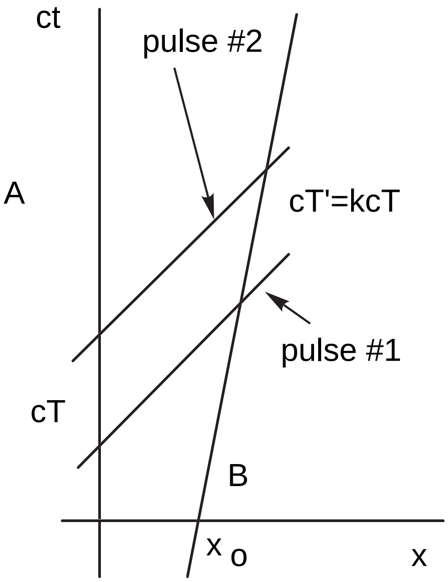


Figure 1

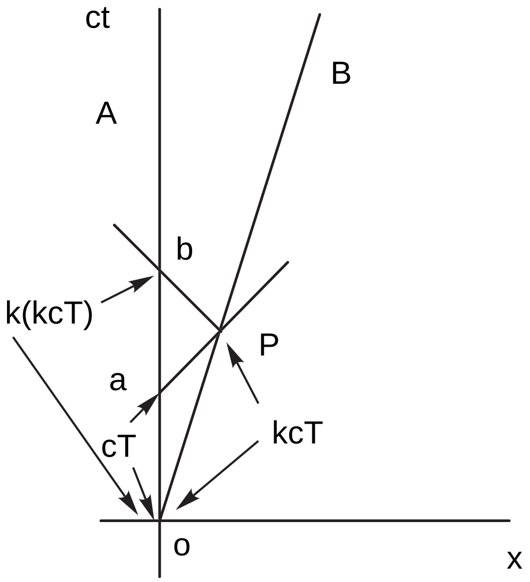


Figure 2

In figure 1, two pulses are sent from A to B. In figure 2, two pulses(one is sent at $t = 0$) are sent from A to B and then B sends each of them back to A.

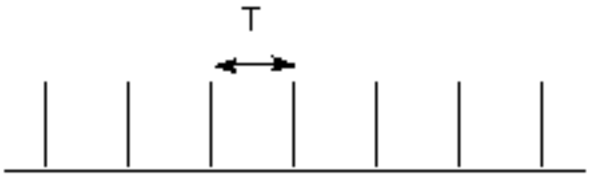
In figure 1, B’s worldline is given by the equation

$$x = x_0 + vt = x_0 + \frac{v}{c}ct \qquad \text{(B is at } x_0 \text{ at } t = 0\text{)}$$

and in figure 2 B’s worldline is given by the equation

$$x = \frac{v}{c}ct \qquad \text{(B is at } x = 0 \text{ at } t = 0\text{)}$$

In both of these cases, we are assuming the light being sent out consists of a series of pulses separated by a time T in the frame of the source (A) as shown below.



For the first experiment, our assumptions say that the interval between reception of the two signals by B (according to a clock that is traveling with B), is cT' and that this interval is proportional to cT (see diagram) with the proportionality factor k that depends only on the relative velocity between A and B, that is,

$$cT' = k(v)cT$$

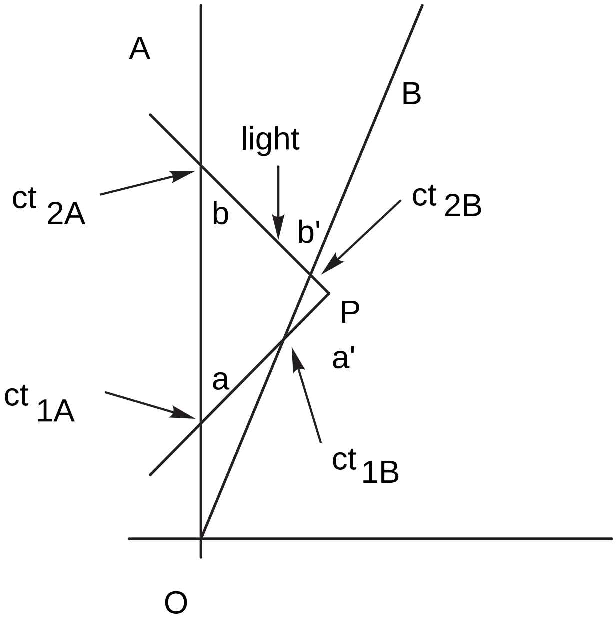
In the second experiment, we see two pulses separated by T sent out by A (the first when they are at same spacetime point) and received by B separated by kT and then sent back to A and received separated by $k(kT)$.

We have used the fact that the physical laws are independent of the relative motion (assumption [1]), which requires that the relationship between A and B be reciprocal, so that, if B emits two signals separated by an interval cT (according to B's clock), then A must receive them with an interval kcT (according to A's clock). Therefore the intervals go like

$$cT \to kcT \to k(kcT)$$

as shown in the diagram.

Now Consider the experiment below:



Here is what is happening in this diagram.

(1) A and B synchronize their clocks to zero when their worldlines cross at event O.
(2) After a time T(according to A) A sends a light signal to P - this is event a (a is on A's worldline).
(3) B receives the light signal at event a'(a' is on B's worldline)
(4) The signal is reflected back to A from event P.
(5) B receives the reflected signal at event b' (b' is on B's worldline).
(6) A receives the reflected signal at event b (b is on A's worldline).

For event P observer A says(using the radar method) that

$$x_P = c\frac{t_{2A} - t_{1A}}{2} \quad , \quad ct_P = c\frac{t_{2A} + t_{1A}}{2}$$

and observer B says(using the radar method) that (same experiment and same equations for both A and B)

$$x'_P = c\frac{t_{2B} - t_{1B}}{2} \quad , \quad ct'_P = c\frac{t_{2B} + t_{1B}}{2}$$

It is clear, using $\Delta x = c\Delta t$ and $\Delta x' = c\Delta t'$ that

$$c(t_{2A} - t_P) = x_P = c(t_P - t_{1A}) \text{ and } c(t_{2B} - t'_P) = x'_P = c(t'_P - t_{1B})$$

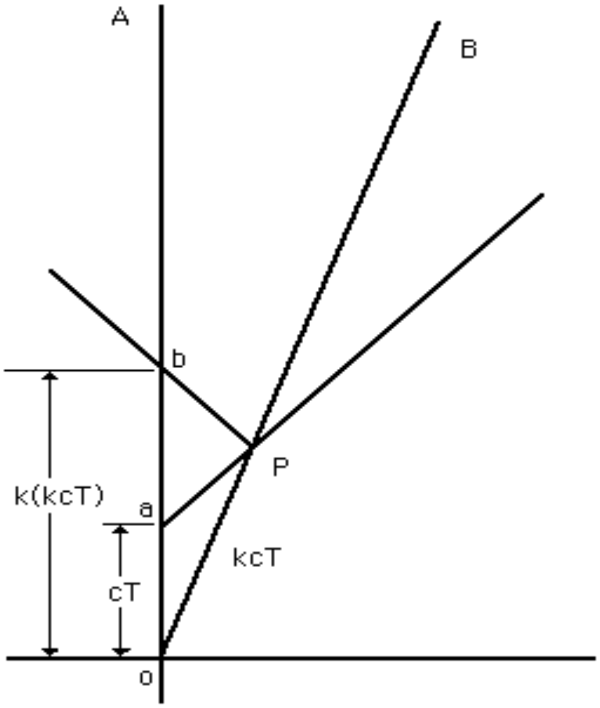
or

$$\begin{aligned} ct_{2A} &= ct_P + x_P \text{ and } & ct_{2B} &= ct'_P + x'_P \\ ct_{1A} &= ct_P - x_P & ct_{1B} &= ct'_P - x'_P \end{aligned}$$

Our earlier experimental results(figure 2) now imply that

$$ct_{1B} = kct_{1A} \text{ and } ct_{2A} = kct_{2B}$$

as shown in the diagram below



i.e., for observer B, the interval $OP = kcT$ (according to B's clock) and for observer A, the interval $Ob = k(kcT)$ (according to A's clock).

Therefore, A has sent out a signal to event P at $ct_{1A} = cT$ and received it back at $ct_{2A} = k^2 cT$

Putting everything together and doing some algebra we get

$$ct'_P + x'_P = \frac{ct_P - x_P}{k}$$
$$ct'_P - x'_P = k(ct_P - x_P)$$

Further algebra then gives(dropping the subscript P since there is nothing special about that particular spacetime point) using the value of k from assumption [3]

$$ct' = \gamma(ct - \beta x) \quad \text{and} \quad x' = \gamma(x - \beta ct)$$

where

$$\beta = \frac{v}{c} \quad \text{and} \quad \gamma = \sqrt{\frac{1}{1-\beta^2}}$$

These are the so-called **Lorentz transformations.** They allow the two observers to relate their experimental results. They are "translators" between experiments done in different frame moving relative to each other with constant velocity in the common $x(x')$ direction.

We note that for relative motion in the x-direction (as above) the y and z coordinates are unchanged,i.e.,

$$y' = y \quad , \quad z' = z$$

So that we have the relations

$$ct' = \gamma(ct - \beta x) \quad , \quad x' = \gamma(x - \beta ct)$$
$$y' = y \quad , \quad z' = z$$

We first note that as $v \to 0$, $k \to 1$ which implies no difference between A and B (which is correct because they will then be at rest relative to each other).

Note the **mixing** of space and time so that neither is any longer independent of the other. **A very dramatic occurrence**.

So the principle of relativity together with two experimental results allows us to derive these new relations which constitute basic equations of the theory of special relativity.

This is the way theoretical physics works.

We take a mixture of general principles (things that no one can argue with) and experimental results and create a set of assumptions about the way the world works. We then derive the consequences of these assumptions, in this case, the Lorentz Transformations.

We then have a theory that agrees with our assumptions (we will show that shortly). If the theory represents a new paradigm in physics then we should be able to make new predictions not related to our assumptions that agree with all future experiments.

We can make the immediate prediction that nothing can travel faster than light. Look at the form of the γ-factor. If it were possible for $v > c$, then one observer could measure two events separated by "real" time and space intervals while a second observer would have to measure "imaginary" intervals. Since this has never been observed to happen, we can confidently predict that all objects must have $v < c$ so that γ is always real. This is corroborated by all known experiments..

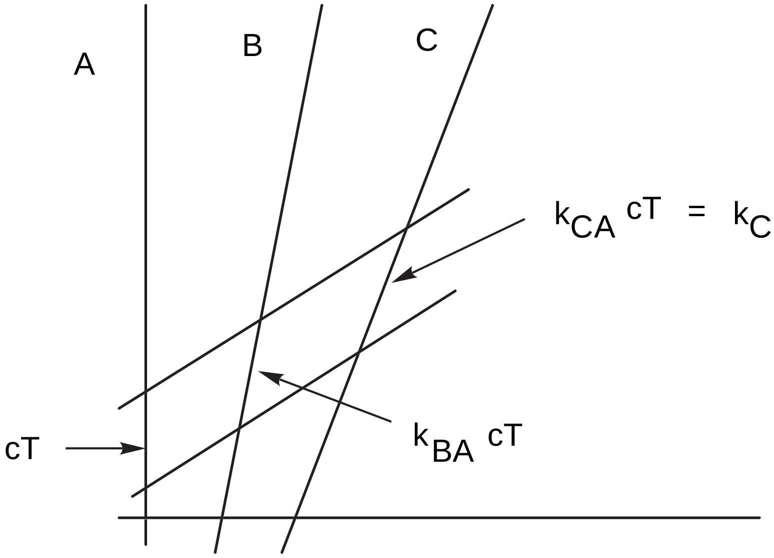
This encourages the theorist to proceed further and see what other interesting features are lurking about.

Features of the Theory

Now that we are confident about our theory, let us work out some other features it predicts.

Suppose that we now have three observers A, B and C, such that velocity of B relative to A is $v_{BA} > 0$ and velocity of C relative to A is $v_{CA} > 0$ and velocity of C relative to B is $v_{CB} > 0$.

A then sends out two light signals, separated by interval cT (according to A) that are received by both B and C (as shown in the diagram below).



We know from previous discussions that B thinks the interval between signals is $k_{BA}cT$ and C thinks it is $k_{CA}cT$, where

$$k_{BA} = \sqrt{\frac{c+v_{BA}}{c-v_{BA}}} \quad , \quad k_{CA} = \sqrt{\frac{c+v_{CA}}{c-v_{CA}}}$$

In a similar manner, C could assume that the signals came from B and not A and therefore would think the interval is $k_{CB}(k_{BA}cT)$, where

$$k_{CB} = \sqrt{\frac{c+v_{CB}}{c-v_{CB}}}$$

But these two results must be identical(according to C) which means we must have

$$k_{CA}cT = k_{CB}(k_{BA}cT)$$
$$k_{CA} = k_{CB}k_{BA}$$

This is the **velocity addition formula**. Converting to velocities we have

$$v_{CA} = \frac{v_{CB}+v_{BA}}{1+\frac{v_{CB}v_{BA}}{c^2}}$$

This reduces back to the Newton-Galileo result for $v \ll c$, as it must, i.e.,

$$v_{CA} = v \quad , \quad v_{CB} = v' \quad , \quad v_{BA} = u$$
$$v_{CA} = \frac{v_{CB}+v_{BA}}{1+\frac{v_{CB}v_{BA}}{c^2}} \to v_{CB}+v_{BA}$$
$$v = v' + u \to v' = v - u$$

Finally, if $v_{CB} = c$ (B is looking at a light signal) and $v_{BA} = u$ (B is moving relative to A), then we find that

$$v_{CA} = \frac{u+c}{1+\frac{uc}{c^2}} = c$$

and we have the prediction (**or verification of our assumption**) that if **one observer** measures something moving with the speed of light c, then **all observers** will also measure its speed to be c.

In this new picture, space and time merge into a new 4-dimensional continuum.

The most important variables in any theory are those that are unchanged for different observers. Such objects are called **invariants**.

The speed of light is such an invariant.

Another invariant is the so-called **spacetime interval**, which is constructed as follows.

Observers A and B can independently measure the spacetime coordinates for two events

$$\begin{aligned}&Observer\quad A:(ct_{A1},x_{A1},y_{A1},z_{A1})\ \text{ and }\ (ct_{A2},x_{A2},y_{A2},z_{A2})\\&Observer\quad B:(ct_{B1},x_{B1},y_{B1},z_{B1})\ \text{ and }\ (ct_{B2},x_{B2},y_{B2},z_{B2})\end{aligned}$$

The Lorentz transformations relate these coordinates by

$$\begin{aligned}&ct_{B1}=\gamma(ct_{A1}-\beta x_{A1})\ ,\ x_{B1}=\gamma(x_{A1}-\beta ct_{A1})\ ,\ y_{B1}=y_{A1}\ \ ,\ \mathrm{z}_{B1}=\mathrm{z}_{A1}\\&ct_{B2}=\gamma(ct_{A2}-\beta x_{A2})\ ,\ x_{B2}=\gamma(x_{A2}-\beta ct_{A2})\ ,\ y_{B2}=y_{A2}\ \ ,\ \mathrm{z}_{B2}=\mathrm{z}_{A2}\end{aligned}$$

Now the spacetime interval for an observer, is **defined** in general for any two events by

$$(\Delta s)^2=c^2(\Delta t)^2-(\Delta x)^2-(\Delta y)^2-(\Delta z)^2$$

It is then easy to show using the Lorentz transformations that the corresponding spacetime intervals for any two observer for the two events above

$$\begin{aligned}(\Delta s_A)^2&=c^2\left(t_{A2}-t_{A1}\right)^2-\left(x_{A2}-x_{A1}\right)^2-\left(y_{A2}-y_{A1}\right)^2-\left(z_{A2}-z_{A1}\right)^2\\(\Delta s_B)^2&=c^2\left(t_{B2}-t_{B1}\right)^2-\left(x_{B2}-x_{B1}\right)^2-\left(y_{B2}-y_{B1}\right)^2-\left(z_{B2}-z_{B1}\right)^2\end{aligned}$$

are invariant, i.e.,

$$(\Delta s_A)^2=(\Delta s_B)^2$$

We will investigate the powerful consequences of this result shortly.

Let me now present an alternative derivation (for the more mathematically and philosophically inclined) of special relativity that illustrates the powerful methods of theoretical physics.

We have the following **postulates (postulates ⇔ theory)**

(1) All the laws of nature (not just mechanics) must be the same for all observers moving with constant velocity relative to each other.

[if we were to write “All the laws of nature must be the same for all observers" (with no qualification about speeds), then we could derive General Relativity(a much harder derivation)].

This is the Principle of Relativity and restricts the **form** of the laws in each frame.

(2) The speed of light is an invariant.

(3) The motion of a particle observed to be linear in one inertial frame must be linear in all inertial frames.

$$x = x_0 + vt \to x' = x'_0 + v'\,t'$$

This implies that the Lorentz transformations must be linear. Our imposition of the red shift experiment in the first derivation is **equivalent** to this postulate.

We now do a **Gedanken or thought experiment.**

We consider two frames K and K' moving relative to each other with speed v. At the instant that the two origins coincide, we set both clocks to zero, i.e., their worldlines cross at the event

$$(x = 0, ct = 0) \quad , \quad (x' = 0, ct' = 0)$$

and a light pulse is emitted. The equations that describe the propagation of the light pulse (a sphere in space where $r = ct$ and $r' = ct'$ (note same c) must be of the same form in each frame (Postulate 1).

We then have

$$c^2t^2 - x^2 - y^2 - z^2 = \Delta s^2 = 0 \quad \text{in K}$$
$$c^2t'^2 - x'^2 - y'^2 - z'^2 = \Delta s'^2 = 0 \quad \text{in K'}$$

We have explicitly used the second postulate at this point (same c).

These equations state that the vanishing of the spacetime interval between two events in any inertial frame implies the vanishing of the interval between the same two events in **any other inertial frame**. However, we want to prove a more powerful statement, namely, that $\Delta s^2 = \Delta s'^2$ in general(not just when it is zero)!

We now use the third postulate. A general theorem from the mathematics of quadratic forms or a lot of messy algebra then says that, under the above conditions, these two quadratic forms Δs^2 *and* $\Delta s'^2$ can be connected, at most, by a proportionality factor $\Delta s'^2 = g(x, y, z, t, \vec{v})\Delta s^2$.

Now all physical theories assume that for a **free particle**

(1) the laws of motion are independent of the choice of origin for the coordinate system

(2) the laws of motion are independent of the orientation of the coordinate system

(3) its velocity during **any time interval** is the same

These are rules that correspond to the statement spacetime is **homogeneous.**

This implies that the proportionality factor can only depend on $\vec{v}$, i.e., $\Delta s'^2 = g(\vec{v})\Delta s^2$

Physicists also assume that space is **isotropic**, which means we cannot have a dependence on the direction of $\vec{v}$. Thus we have $\Delta s'^2 = g(v)\Delta s^2$ where v = the magnitude of $\vec{v}$.

Now, if we transform from K' back to K we must have the inverse result $\Delta s^2 = g(v)\Delta s'^2$ since $-\vec{v}$ has the same magnitude as $\vec{v}$.

Putting these two results together we have $g^2 = 1 \rightarrow g = \pm 1$ are the only possibilities. g is a constant, but which one?

If we let $v \rightarrow 0$ then the systems K and K' become identical and hence $g(0) = 1$ and since g is a constant, we must have $g(v)$ = +1.

We have thus proved that $\Delta s^2 = \Delta s'^2$ in general.

Once we have invariance of the spacetime interval and the linearity of the transformation equations between frames it is straightforward to derive the Lorentz transformations and all the other results follow, which is what your textbook (Moore)does.

Minkowski Diagrams

We can visualize the Lorentz transformation by superposing the (x',ct') and (x,ct) planes into a **common** diagram called a **Minkowski or spacetime diagram** by following these steps:

[1] Choose the (x,ct) axes to be perpendicular(we are always free to do this for one set of axes).
[2] Calibrate these axes(arbitrary choice).
[3] Locate the x' and ct' axes within the framework of the (x,ct) axes.

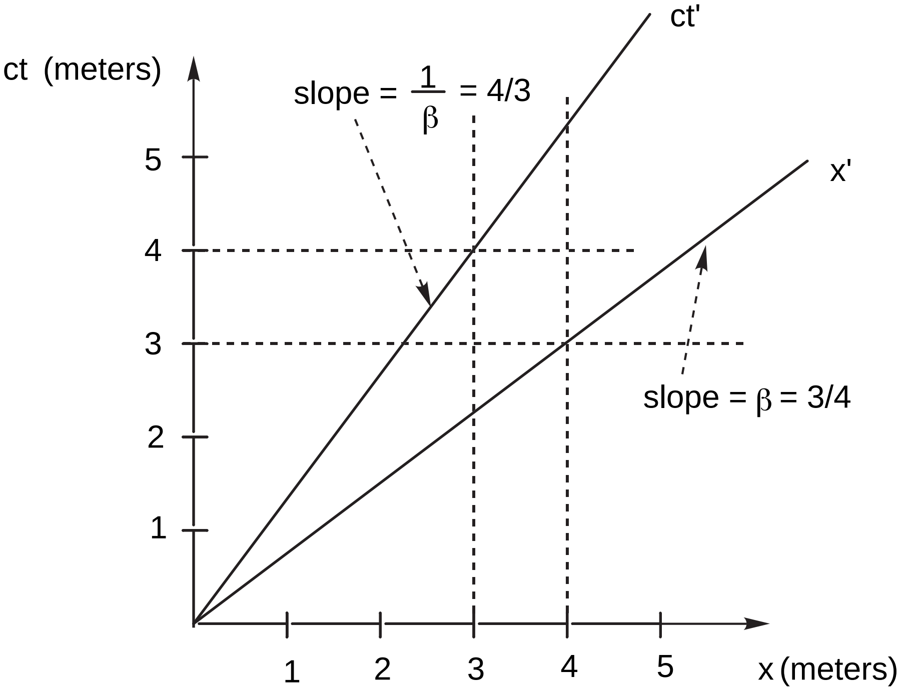
The x' axis is the line $ct' = 0$ and the ct' axis is the line $x' = 0$.

From the Lorentz transformations these lines correspond to the equations:

$$x' = \gamma(x - \beta ct) = 0 \rightarrow ct = \frac{1}{\beta}x \rightarrow ct' - axis$$

$$ct' = \gamma(ct - \beta x) = 0 \rightarrow ct = \beta x \rightarrow x' - axis$$

Thus, the x'-axis is a straight line with slope $\frac{1}{\beta}$ in the (x, ct) plane and the ct'-axis is a straight line with slope β in the (x, ct) plane as shown in the diagram below for the case $\beta = 3/4$:



Thus, we can only choose one set of axes as perpendicular!! We have no choice for the second set of axes if we want them to coexist on the same diagram!!

Now we see why we had to make the correct choice about parallel versus perpendicular for determining the coordinates of an event. They give very different results for non-perpendicular axes.

[4] Calibrate the primed axes using the invariance of the interval as follows:

Consider two events, namely $(0,0)$ and (x, ct) such that

$$(\Delta S)^2 = c^2(\Delta t)^2 - (\Delta x)^2 = c^2t^2 - x^2 = -1$$

For the second observer, these events are (0, 0) and (x', ct') such that

$$(\Delta S')^2 = c^2(\Delta t')^2 - (\Delta x')^2 = c^2t'^2 - x'^2 = -1$$