

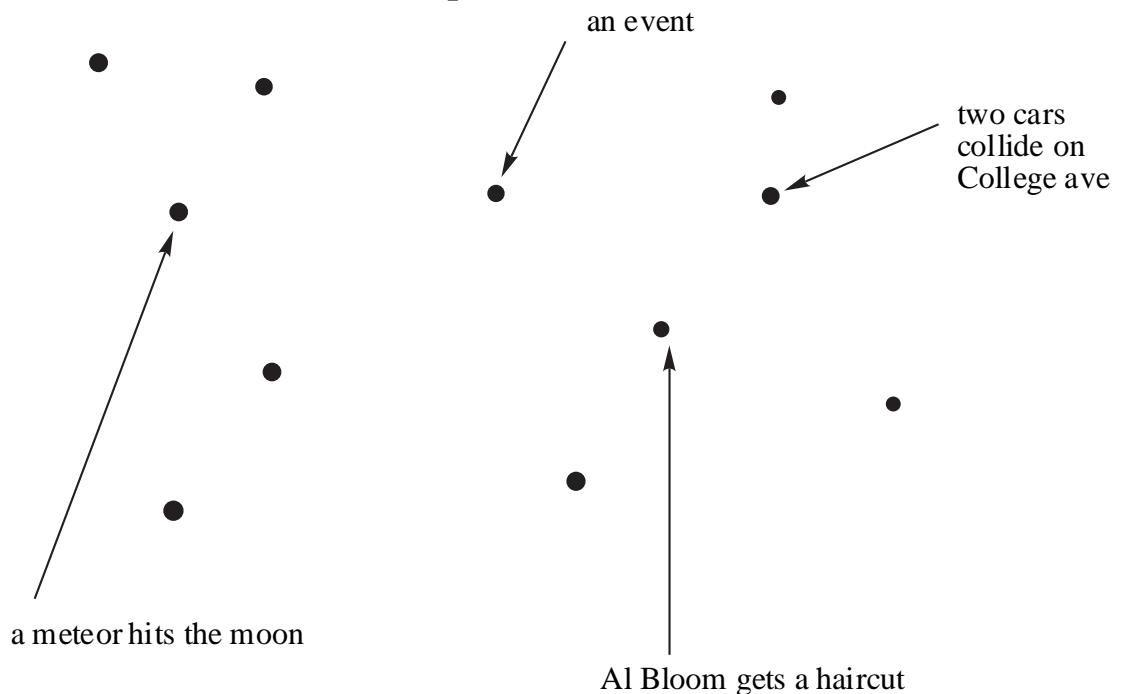
## General Ideas about Spacetime

### Definitions

An **event** intuitively means something happening in a fairly limited region of space and for a short duration in time. Mathematically, we idealize this concept to become a point in space and an instant in time. In the universe, as we understand it at this time, it requires 4 numbers to specify an event, namely, three numbers to describe spatial position and one number to describe time. This is called 4-dimensional spacetime. We will modify this later on.

Everything that happens in the universe is either an event or a collection of events. Events are independent of observers. The four numbers describing an event are not, as we shall see, independent of observers.

**Spacetime** is the collection of **all possible events**.



How do we measure the "coordinates" of an event?

One method is the so-called many-observer model. It works as follows:

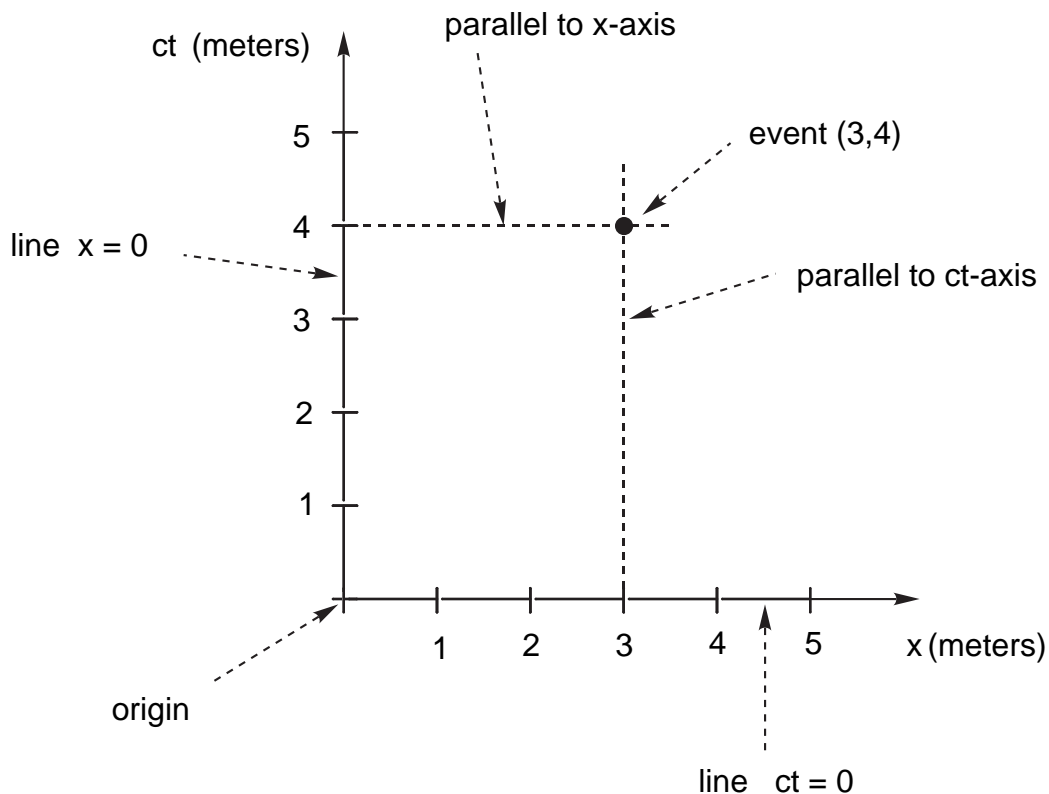
- (1) synchronize clocks ahead
- (2) measure and label grid locations ahead
- (3) observers move clocks to grid locations (assume has no effect on synchronization)
- (4) throw eraser into the air
- (5) if eraser passes an observer's location then observer records local time
- (6) collection of such "where and when" information --> set of events representing motion being observed

This "operational definition" of each event is simply one possible prescription for assigning numbers to the associated where and when in a precise and reproducible way.

For our purposes, we will assume a two-dimensional spacetime consisting of one spatial dimension and one time dimension. All the physics that we derive in this restricted universe is easily extended to the real 4-dimensional universe.

A particular set of coordinate axes and associated scales are chosen inside spacetime at our convenience and only so that we can relate the events to measured quantities in experiments, i.e., so that the theorists can talk to the experimentalists.

We represent events using a **spacetime diagram** (as shown below in the 2-dimensional case). Note that we use  $ct$  rather than  $t$  for the vertical axis, where  $c$  is the speed of light ( $c = 3.0 \times 10^8$  m/sec). This is just a change in scale for the vertical axis and the reason for this will become clear later.



Note this is a parallel line definition of coordinate values rather than a perpendicular definition (they are different as we shall see). The definition of the coordinate axes is given by

$ct$ -axis is the line  $x = 0$  and  $x$ -axis is the line  $ct = 0$

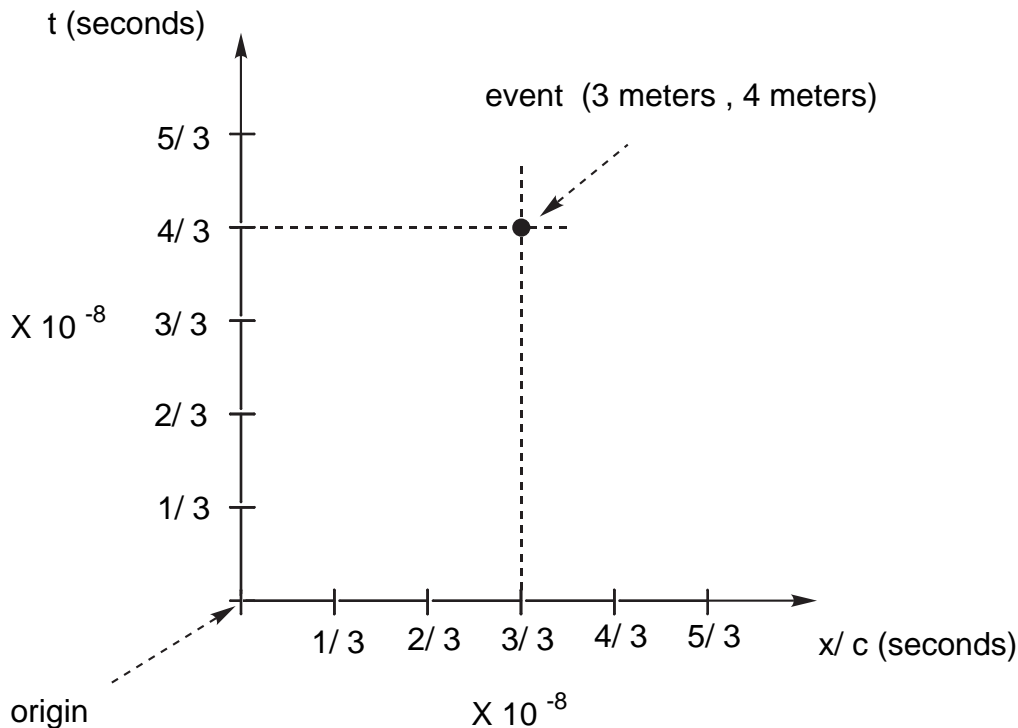
The vertical scale is really **time** but rescaled by the speed of light, i.e.,

$$ct = 1 \text{ meter} \rightarrow t = \frac{1 \text{ meter}}{3 \times 10^8 \text{ meter/sec}} = \frac{1}{3} \times 10^{-8} \text{ sec} = 3.33 \text{ nanosecond} = 3.33 \text{ ns}$$

Note that your textbook uses a different scheme for labelling the axes as shown below:

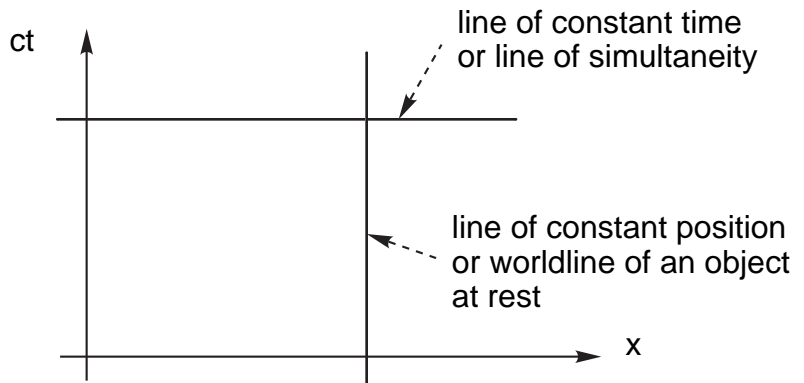
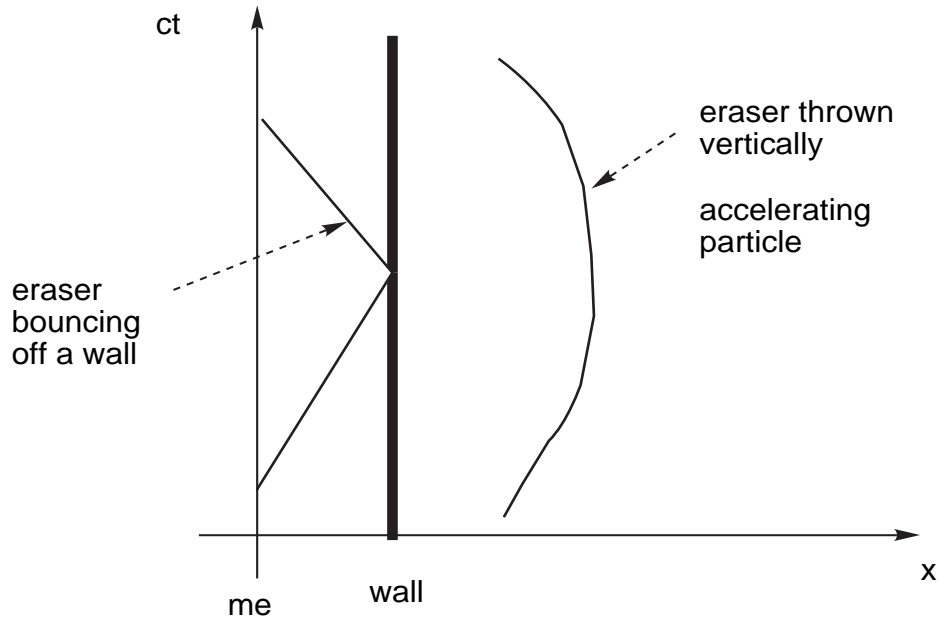
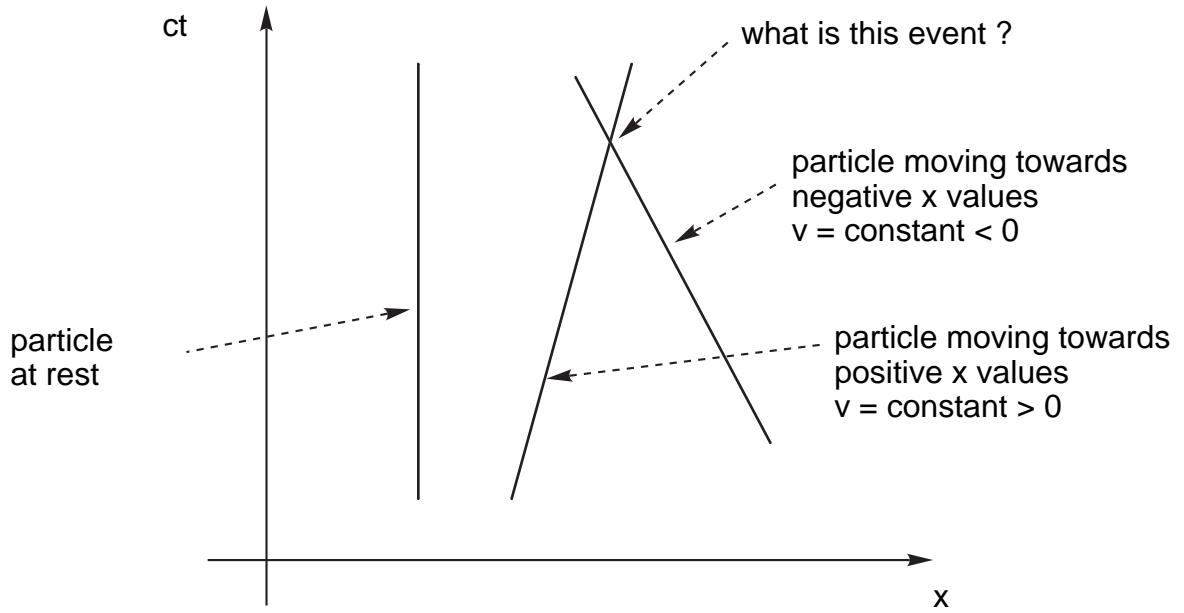
both  $t$  and  $x$  are measured in seconds instead of meters

so that  $ct = 1 \text{ meter} \rightarrow t = \frac{1}{3} \times 10^{-8} \text{ second}$  and  $x = 1 \text{ meter} \rightarrow \frac{x}{c} = \frac{1}{3} \times 10^{-8} \text{ second}$  as shown below.

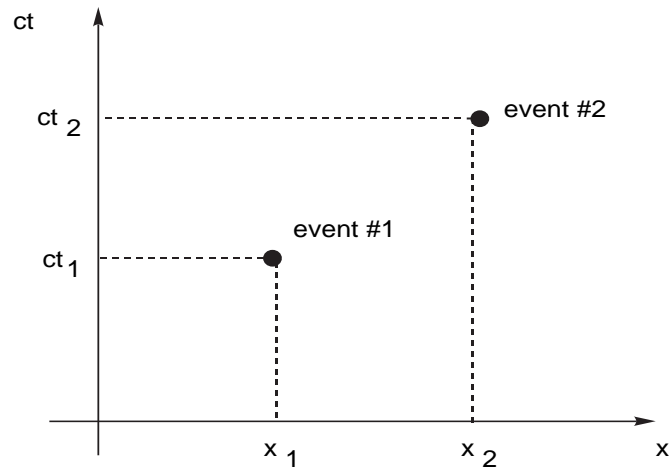


I think it is better to use  $x$  and  $ct$  axes as we shall see. Since both of these schemes are extensively used in physics it is best that you to understand both.

A collection of related events is called a **worldline**. Some examples of worldlines and other things are shown below:



Let us now discuss these concepts in more detail. We start with things from everyday experience, which is something we hope that we know something about! Consider the diagram below



The quantity  $\Delta t = t_2 - t_1$  is called the **time-separation** or **coordinate time** between the events and the quantity  $\Delta x = x_2 - x_1$  is called the **spatial-separation** between the events.

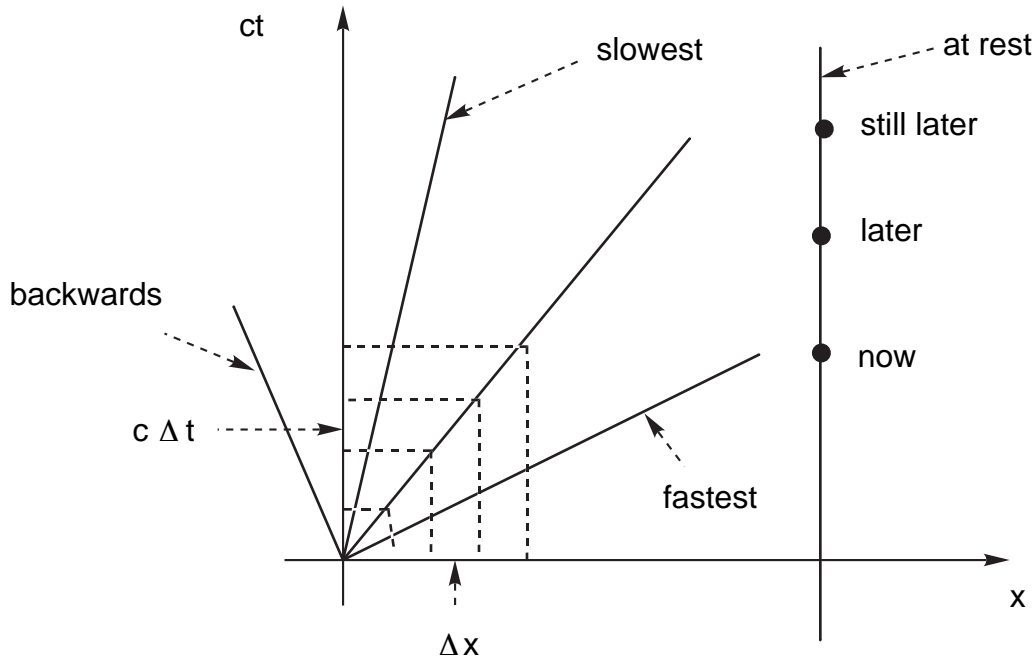
Can we also say at this point that  $\Delta t$  = time-interval between events and  $\Delta x$  = distance between events?

**The answer is NO!**

We must be very careful not to make any such assumptions when we cannot prove the statements; a good rule is -- **if we do know something is true, then we should not assume it!!**

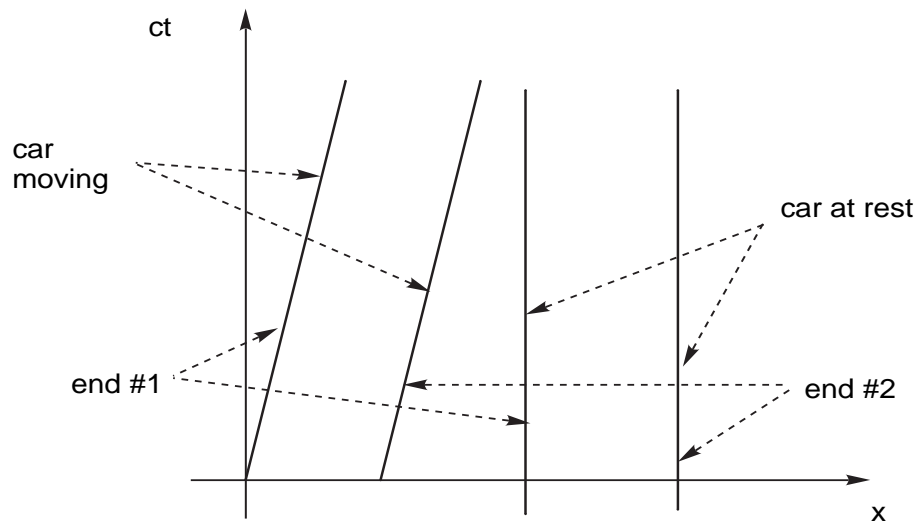
It is clear, however, that two events on the same vertical line take place at the same position and two events on the same horizontal line take place at the same time (they are **simultaneous**).

In the diagram below we have several objects all moving with different speeds.



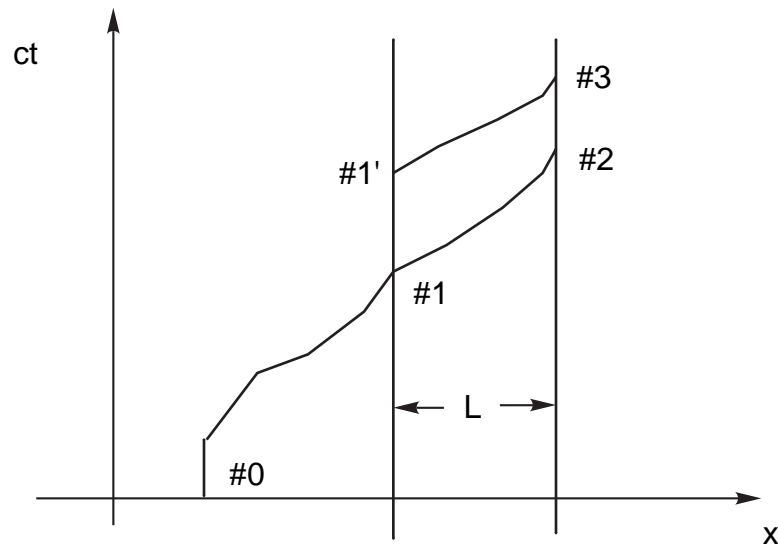
In all cases, during any interval the speed is  $v = \frac{\Delta x}{\Delta t} = 1/\text{slope}$ .

Now let us imagine a car on a track and create a diagram to represent the motion of the car. The diagram below shows the worldlines corresponding to a car at rest and a car moving with constant speed in the positive x-direction.



Now let us attempt to measure the length of the car. First we consider the car at rest. The diagram below represents me walking (in your frame of reference) first to one end of the car and recording its position( $x_1$ ) and then walking to the other end and recording the position( $x_2$ ). Where are you on this diagram?

**You are the ct-axis in your own frame of reference!!**



At event #0, I am standing at rest and talking. I then walk over to one end of the car (event #1 =  $(x_1, ct_1)$ ). I then walk over to the other end of the car (event #2 =  $(x_2, ct_2)$ ). Alternatively, I could have delayed walking over to the other end of the car (event #1' =  $(x_1, ct_1)$ ) and then gone over to the other end (event #3 =  $(x_3, ct_3)$ )

The length of the car is then  $L = x_2 - x_1$  or  $L = x_3 - x_1 = x_2 - x_1$ .

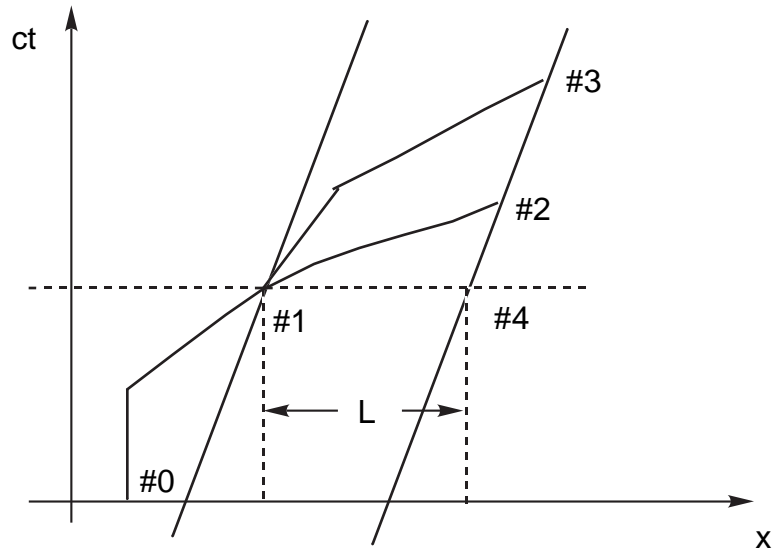
For a car at rest, the length measurement is the same no matter how long I delay (or whether I use event #2 or #3).

Notice how the car is just in spacetime. We do not have to be there!!. What is actually in spacetime for the car? Look carefully.... all of its past, all of its future --> everything about the car is in spacetime!!!!

What is the difference if we use our many-observer model? Two observers are located at the ends of the car. They record the locations and we then calculate the length. There are no gains with this approach for a car at rest!!

At this point we do not seem to have any problems with a length measurement.

Now consider a moving car as show below:



At event #0, I am standing at rest and talking. I then walk over to one end of the car (event #1 =  $(x_1, ct_1)$ ). I then walk over to the other end of the car (event #2 =  $(x_2, ct_2)$ ). Alternatively, I could have walked over more slowly to the other end of the car (event #3 =  $(x_3, ct_3)$ ).

In this case,  $x_3 \neq x_2$ . Is the length of the car  $L_{12} = x_2 - x_1$  or  $L_{13} = x_3 - x_1 > L_{12}$ ?

As can be seen from the diagram, neither is the correct result  $L$ .

Is there an operational procedure that we can use to guarantee that we will always measure the correct length (defined to be the length measured at rest)?

The diagram indicates the answer. If we measure the location of the ends of the car at the same time (simultaneously), namely, events #1 and #4, then we get the correct length  $L$  (or along any other line of simultaneity).

Of course, this is an impossible measurement for a single observer, but not for the many-observer model. We just have all observers close their eyes and when their clock alarms go off (all set to go off simultaneously), then two of the observers will be located at the ends of the car (even if it is moving) and the length is the spatial separation of their grid locations.

So we "define" a length measurement as

**the spatial separation between the endpoints of the object measured simultaneously**

Philosophers would not let me use the word "define" for this "operational procedure", but they are not here now to challenge me!

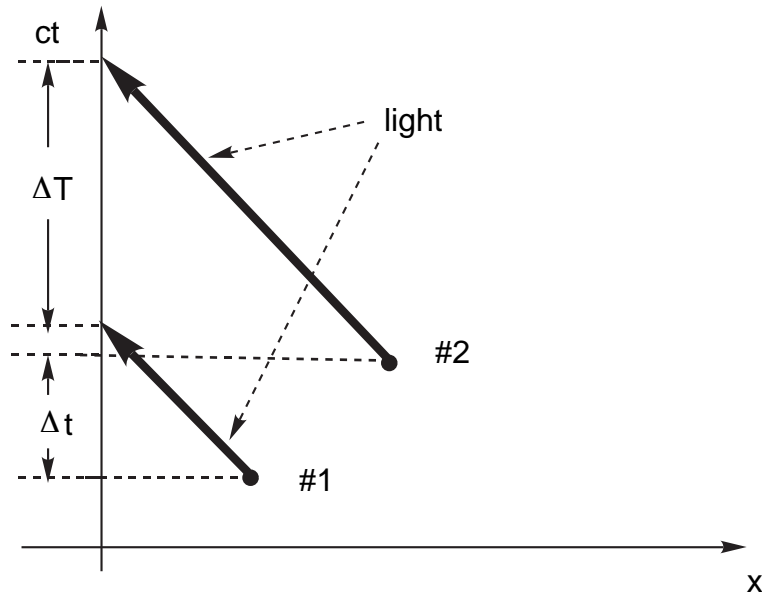
**A question:** Have we just exchanged the unknown meaning of "length" for a new unknown "simultaneity" ?

The answer is YES!

However, that is what operational procedures are all about and that is why they differ from "definitions".

We think, at this point, that we will be able to define simultaneity unambiguously and that is better than not knowing how to "measure" length.

What about time intervals? Consider the two events shown below:



The time-separation between events #1 and #2 is  $\Delta t = t_2 - t_1$

Now my worldline is the  $ct$ -axis. Can I measure this quantity? Remember, I can only have confidence in measuring instruments that are always on my worldline (i.e., always with me). The answer is NO!.

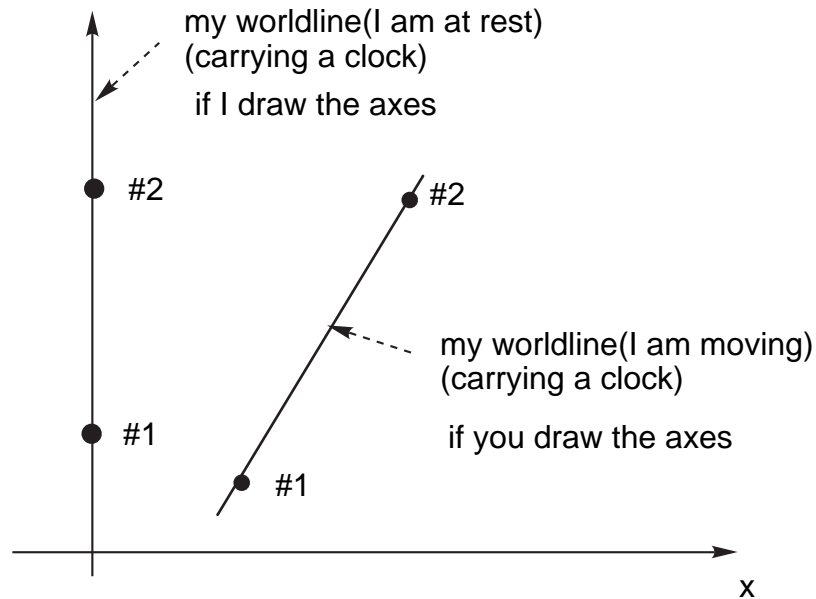
I can, however, measure the quantity  $\Delta T = \left(t_2 + \frac{x_2}{c}\right) - \left(t_1 + \frac{x_1}{c}\right) = \Delta t + \frac{\Delta x}{c}$  where  $\Delta x = x_2 - x_1$  since that measurement can be made with a clock I am carrying with me on my worldline.

It is clear that  $\Delta T > \Delta t$ . If I independently know the spatial separation between the two events, then I could infer (calculate) the time-separation, but this is not a measurement!

$\Delta T$  is the time I "see" between the two events.

Is either of these the time interval? We just do not know!

We must create an operational definition for the time interval. This is done as follows:



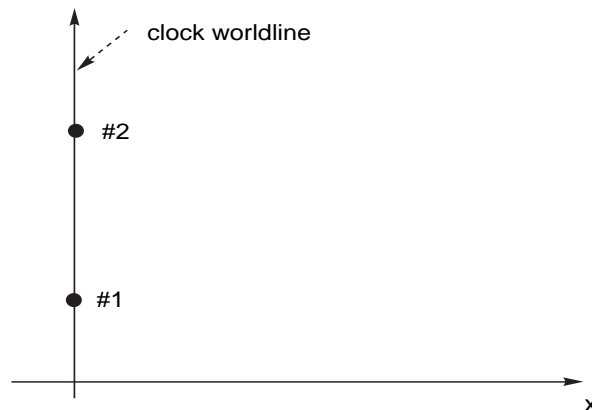
In both cases, the **time interval** is operationally defined as the difference in my clock readings, i.e., the clock and thus me must have a worldline that passes through **both** events in order to define the time-interval between the events in an unambiguous manner.

This prescription assumes that nothing happens to a clock when it moves that changes this result. We do not know that this is true.

Thus, to be safe, we "define"

**time interval between two events = time-separation when clock is at rest and thus, the two events take place at the same position according to the observer carrying the clock**

as shown below:



The two events in this case are

$$\begin{aligned}\text{event \#1} &= (x, ct_1) \\ \text{event \#2} &= (x, ct_2)\end{aligned}$$

and the time interval or elapsed time between events is given by  $t_2 - t_1$ .

Is this what you actually do? NO.

You move between events usually (changing your speed in the process) and assume that this has no effect on the clock or you stay still and infer the time interval by measuring the time that "see" between the two events. As we shall see from the theory we are developing, this is OK for the everyday world we live in but not in a world where objects move with large speeds.

We are assuming that all of these measurement procedures are objective. Suppose there is a rotten core in the apple of scientific objectivity. Physics as we shall present it works .... it makes correct predictions. Does it matter if we are really being subjective, i.e., that our entire view of spacetime might be dependent on human observation or that all measurements are "relative". Just food for thought at this point. More about this later when we study quantum physics.`

So we now have operational definitions that allows a **single observer** looking at the universe to describe events, measure distances and time intervals between events, and so on and report on what happened in some experiment.

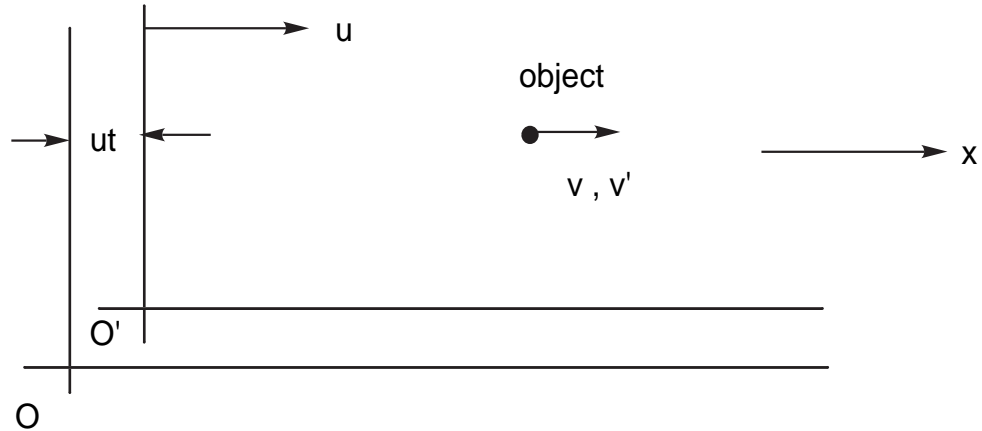
Our problem arises, however, when a second observer, who is moving relative to the first observer, appears and also tries to describe the experiment using the same procedures.

### **Galilean Relativity**

Central to any discussion of the relativity that prevailed alongside Newtonian (pre-Einstein) physics is the concept of **absolute time**.

Newton and Galileo assumed that the passage of time was the **same for all observers** no matter what they were doing. Thus if two observers separately measured the time interval between two events, then it was assumed that  $\Delta t = t_2 - t_1 = t'_2 - t'_1 = \Delta t'$ .

Suppose that two observers are moving with respect to each other (along a common  $x$ -direction) with relative speed  $u$  such that their respective origins coincide at  $t = t' = 0$ . Then, at some time  $t$  later we might have the situation shown below.



We know from **everyday experience** that if observer O' measures a velocity  $v'$  and observer O measures a velocity  $v$  for some object, the relationship between these two measured velocities is given by  $v' = v - u$ .

Now to measure the velocity(constant) of an object, each observer must observe two events in its motion. Suppose that has occurred and we have the measured results for the two events:

$$\#1 \rightarrow (x_1, ct_1) \text{ and } (x'_1, ct'_1)$$

$$\#2 \rightarrow (x_2, ct_2) \text{ and } (x'_2, ct'_2)$$

We then have

$$v = \text{velocity measured by S} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$$

$$v' = \text{velocity measured by S'} = \frac{x'_2 - x'_1}{t'_2 - t'_1} = \frac{\Delta x'}{\Delta t'}$$

Now the absolute time concept says that  $\Delta t = \Delta t'$  and this then implies that

$$v' = v - u$$

$$\frac{\Delta x'}{\Delta t'} = \frac{\Delta x'}{\Delta t} = \frac{\Delta x}{\Delta t} - u$$

or

$$\Delta x' = \Delta x - u\Delta t$$

Now if we choose the events representing the measurement of the particle velocity to be

$$\text{Event \#1} - (x = 0, t = 0), (x' = 0, t' = 0)$$

$$\text{Event \#2} - (x = x, t = t), (x' = x', t' = t)$$

which is just a choice of origin values for space and time measurements (always allowed because physical phenomena are not dependent on choice of origin), we then obtain the equations

$$ct' = ct$$

$$x' = x - ut = x - \left(\frac{u}{c}\right)ct$$

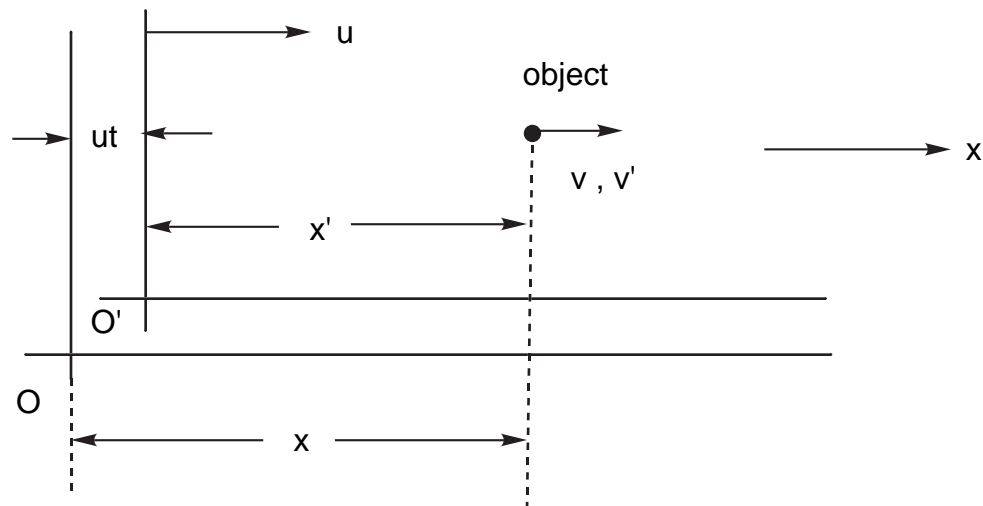
or

$$c\Delta t' = c\Delta t$$

$$\Delta x' = \Delta x - u\Delta t = \Delta x - \left(\frac{u}{c}\right)c\Delta t$$

as the equations **relating the two sets of observations.**

This relationship is shown below:



These are called the equations of the **Galilean transformation.**

They allow two observers in frames of reference moving with constant speed relative to each other to compare their respective observations under the assumption that Newtonian/Galilean physics is valid.

Galilean relativity was the basis of Newtonian physics until 1900. You have a deep understanding of Galilean relativity ingrained within your brain. If you did not, then you would not have survived to be attending Swarthmore College. Galilean relativity accurately describes the everyday world we live in.

**The relative velocity formula is one of the signatures of the "old" classical physics and the everyday world.**

This is the "Old Original World View" of 19th century physics circa 1900 .... a product of the finest minds developed over several centuries. Everyone was comfortable with the theory. It was

internally consistent. It worked amazingly well (agreed with all experiments).

### **And then there was light..... and Special Relativity**

First, what went wrong?

When measured by an observer at rest relative to the experiment, the speed of light is  $c = 3.0 \times 10^8$  m/s = 186,000 mi/s, which is very large compared to everyday speeds. What is the fastest we have launched any object?

Now if you measure that I can throw an object with a speed of 20 m/s when I am at rest relative to you, then what speed will you observe me throwing it if I am running at a speed of 10 m/s relative to you?

The Galilean velocity addition formula tells us the answer is

$$20 + 10 = 30 \text{ m/s}$$

Suppose instead that I am at rest and throw the object at 20 m/s and you are running in the direction opposite to that of the moving object (towards me) at 10 m/s. What speed will you measure?

Again, the Galilean velocity addition formula tells us the answer  $20 + 10 = 30$  m/s.

Finally, suppose instead that I am at rest and throw the object at 20 m/s and you are running in the same direction as the moving object (away from me) at 10 m/s. What speed will you measure?

Again, the Galilean velocity addition formula tells us the answer  $20 - 10 = 10$  m/s.

So it is clear that in our everyday experience with objects moving at everyday speeds, that Galilean relativity works (or that classical theory is valid). **The observed speed of objects depends on the motion of the source and observer of the object.**

Michelson and Morley did an experiment of this sort with light. They found that the speed of light was always measured to be  $c = 3.0 \times 10^8$  m/s no matter what the source or observer of the light was doing!

Their experiments gave the result that:

the speed of light = constant =  $c$   
**independent of source or observer**

This leads to a direct breakdown of Galilean relativity since Galilean relativity says that for two observers in relative motion

both looking at light we must have  $c' = c - u \neq c$ .

Clearly, a **new theory** was needed. A very careful experiment was forcing us to make a paradigm shift in our theoretical understanding of the world.

We will derive this new theory assuming one **general principle**

[1] The **Principle of Relativity**

**the laws of physics are identical for all  
observers in uniform relative motion**

and the **results of two experiments:**

[2] The speed of light is a universal constant( $c$ ) independent of the motion of source or observer

and

[3] It has been experimentally observed that when a **source** of light and a **detector** of light are moving relative to each other with a speed  $v$  the wavelength of the observed light changes with the relative speed. The experimental result is given by the formula  $\lambda_{observed} = k(v)\lambda_0$  where

$$\lambda_0 = \text{observed wavelength when } v = 0, \quad k(v) = \sqrt{\frac{c+v}{c-v}}, \quad c = \text{speed of light}$$

$v > 0 \rightarrow$  source and observer moving away from each other

This is the famous **galactic red shift** observed by astronomers for light received on the earth from distant galaxies moving away from the earth.

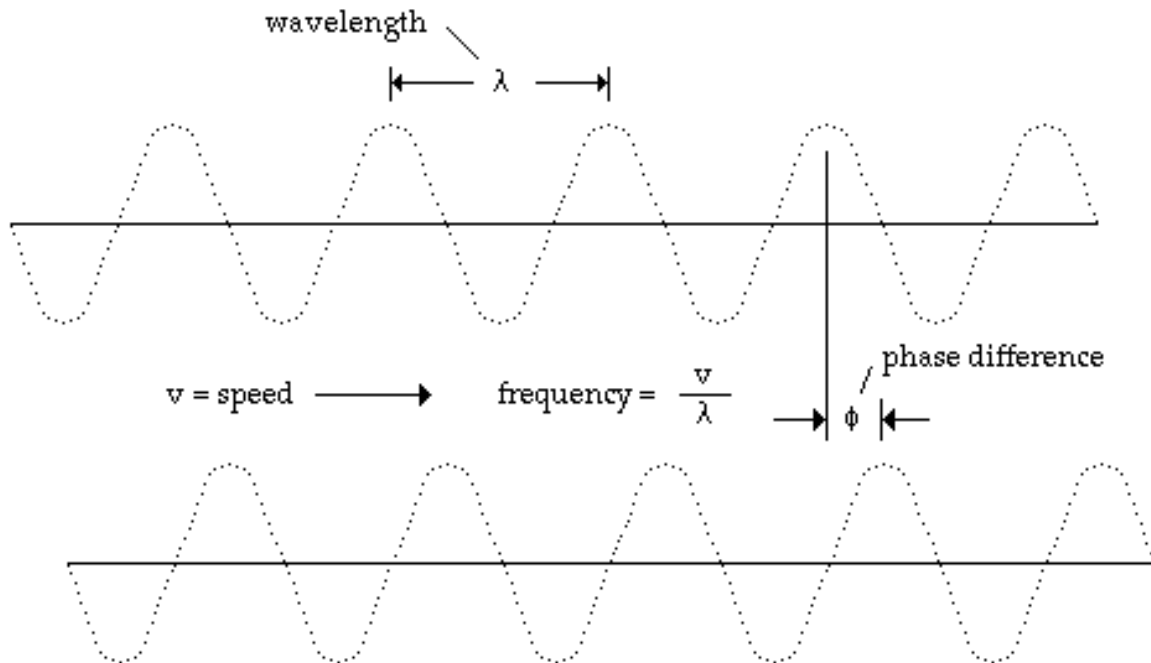
The wavelength of the light is related to frequency and the period by the formula

$$\lambda f = c = \frac{\lambda}{T} \quad f = \text{frequency}, T = \text{period}$$

Other physicists might derive these results with a smaller number of assumptions. For clarity, however, at the level we are working, the derivations will be clearer if we use an extra experimental result. With a lot more work we could do the same derivation leaving out [3].

### **Review of Wave Properties**

**Waves are periodic phenomena in space and time.** A sinusoidal wave illustrates a typical wave ... but we really **only need periodicity.**



**Wavelength** = distance between like points

**Frequency** = time for a point to repeat

**Amplitude** = maximum displacement. Wave energy (intensity) is related to square of amplitude. The **energy content** of a classical wave is proportional to  $A^2$  and is independent of frequency.

**Example:**

$$y = A \cos(kx - \omega t)$$

$$k = \frac{2\pi}{\lambda} \quad , \quad \omega = 2\pi f = \frac{2\pi}{T}$$

$$y = A \cos\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right) = A \cos 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)$$

$$v = \text{wave speed} = \lambda f$$

Fix  $t$   $\rightarrow$  photograph of waveform in space  $\rightarrow$  wavelength

Fix  $x$   $\rightarrow$  oscillation in time  $\rightarrow$  frequency or period

### Interference between Waves

Consider two waves (same amplitude, same frequency, same wavelength) which start out at the same time and propagate in this room. We assume that they travel over different paths and eventually arrive at the same point say on a screen (we assume that the time of arrival is  $t=0$  for simplicity). We then have two waves arriving at the same point with different amplitudes given by

$$y_1 = A \cos kx_1 \quad , \quad y_2 = A \cos kx_2$$

The effect of the waves at this point is given by the sum of the wave amplitudes (that is the way nature works).

$$y = y_1 + y_2 = A \cos kx_1 + A \cos kx_2 = \text{total amplitude}$$

The quantity  $kx$  for each wave is called the **phase**.

We can visualize what happens at the point on the screen by looking at a different experiment. Suppose that we have two waves both traveling along the same line with different starting points. The two traveling waves and their sum look like

$$x_1 = x \quad \text{and} \quad x_2 = x + \text{delta}$$

we have

$$y_1 = A \cos kx \quad y_2 = A \cos k(x + \text{delta})$$

$$y = y_1 + y_2 = A \cos kx + A \cos k(x + \text{delta})$$

So if the waves are in phase (max to max and min to min), which means that they have traveled the same distance ( $\text{delta} = 0$ ) or

$$x_1 = x_2 = x$$

$$y_1 = A \cos kx \quad , \quad y_2 = A \cos kx$$

$$y_1 + y_2 = 2A \cos kx$$

In the same way, if the distances differ by an integral number of wavelengths ( $\text{delta} = n\lambda$ ) we have

$$x_1 = x \quad , \quad x_2 = x + n\lambda = x + 2\pi n / k$$

$$y_1 = A \cos kx \quad , \quad y_2 = A \cos k(x + 2\pi n / k) = A \cos(kx + 2\pi n) = A \cos(kx)$$

$$y_1 + y_2 = 2A \cos kx$$

and we get a large amplitude (brighter spot) in both cases.

But if the waves get out of phase (if the path lengths do not differ by an integral number of wavelengths or zero) then we get smaller total amplitudes and less bright spots.

In particular, if the path difference is exactly  $1/2$  wavelength then the waves cancel, that is, we have

$$x_1 = x \quad , \quad x_2 = x + \frac{1}{2}\lambda = x + \pi / k$$

$$y_1 = A \cos kx \quad , \quad y_2 = A \cos k(x + \pi / k) = A \cos(kx + \pi) = -A \cos(kx)$$

$$y_1 + y_2 = 0$$

When we add waves, it turns out to be just a simple algebraic sum of

their amplitudes at each space-time point

This is called the **principle of superposition**.

As we shall see, this superposition principle will be a universal principle and will dominate much of our later discussions of quantum states (although we will not call them waves...they will, however, be completely equivalent mathematical objects)

**Interference types:**

constructive	phase difference = 0 peaks line up with peaks
destructive	phase difference = 1/2 wavelength peaks line up with valleys

Mathematically this looks like

$$y = A \cos kx + A \cos k(x + d)$$
$$= 2A \cos k \left( x + \frac{d}{2} \right) \cos \frac{kd}{2}$$

Now

$$\frac{kd}{2} = \pi \frac{d}{\lambda}$$
$$d = \lambda \rightarrow \frac{kd}{2} = \pi \rightarrow \cos \frac{kd}{2} = -1 \rightarrow \text{maximum}$$
$$d = \frac{\lambda}{2} \rightarrow \frac{kd}{2} = \frac{\pi}{2} \rightarrow \cos \frac{kd}{2} = 0 \rightarrow \text{minimum}$$

Therefore, we can write

$$c = \frac{\lambda}{T} \quad (\text{for observer at rest wrt source})$$

$$c' = c = \frac{\lambda'}{T'} \quad (\text{for observer moving wrt source})$$

$$T' = T_{\text{observer moving wrt source}} = k(v)T = k(v)T_0 = k(v)T_{\text{observer at rest wrt source}}$$

Here we have explicitly assumed the result of experiment [2] and experiment [3] in writing this formula, namely, that  $c$  = speed of light = constant for all observers and the red-shift relation between time intervals. Using these results as our theoretical assumptions, we can now derive special relativity.