Physics 6H   Freshman Seminar   Fall 2004   Assignment #7

Readings:

- Albert - Chapter 1
- Albert - Chapter 2
- Boccio - Superposition A
- Boccio - Mathematics

Summary: This week we will

- Extend our knowledge about the mathematical language of quantum mechanics
- Learn about color, hardness and superposition

Everyone Problems:

EP-11 Vectors
EP-12 More Vectors
EP-13 Bases
EP-14 More Vectors
EP-23 Eigenvectors
EP-25 Multiply Matrices

Individual Problems:

EP-15 Brackets
EP-16 Soft State
EP-17 Operators Acting
EP-18 Eigenvectors
EP-19 Color Operator
EP-20 Linear Functional
EP-22 Bases and Matrices
EP-24 Spectral Decomposition
EP-26 Matrix Properties

Presentations:

- The Strange World of Color and Hardness
- Dirac Language
    - Kets, Bras and Operators
    - Eigenvalues and Eigenvectors
    - Expectation Values
    - Projection Operators
- Using Matrices Instead
- Gram-Schmidt Process
- Functions of Operators
- Dirac Delta-Function

Seminar Break:

Extra Problems:

EP-11. Vectors - A vector 3 inches long points due west. What is its projection on the axis that points:

(a) due west
(b) due east
(c) due north
(d) straight up

(e) half-way between straight up and due west
(f) half-way between due east and due north
(g) half-way between straight up and due north.

EP-12. More Vectors - Given two vectors

$$\vec{A} = 7\hat{e}_1 + 6\hat{e}_2 - 13\hat{e}_3 \quad , \quad \vec{B} = -2\hat{e}_1 + 16\hat{e}_2 + 4\hat{e}_3$$

(a) Determine $\vec{A} \pm \vec{B}$
(b) Determine $\vec{A} \cdot \vec{B}$
(c) Determine a unit vector in the same direction as $\vec{A}$

EP-13. Bases - Given two vectors $\vec{A} = 7\hat{e}_1 + 6\hat{e}_2$, $\vec{B} = -2\hat{e}_1 + 16\hat{e}_2$ written in the $\{\hat{e}_1, \hat{e}_2\}$ basis set and given another basis set

$$\hat{e}_q = \frac{1}{2}\hat{e}_1 + \frac{\sqrt{3}}{2}\hat{e}_2 \quad , \quad \hat{e}_p = -\frac{\sqrt{3}}{2}\hat{e}_1 + \frac{1}{2}\hat{e}_2$$

(a) Show that $\hat{e}_q$ and $\hat{e}_p$ are orthonormal
(b) Draw a diagram showing $\hat{e}_q$, $\hat{e}_p$, $\vec{A}$ and $\vec{B}$.
(c) Determine the new components of $\vec{A}$ and $\vec{B}$ in the $\{\hat{e}_q, \hat{e}_p\}$ basis set

EP-14. More Vectors - For the points A(3,1,1), B(4,2,0) and C(0,4,3) answer the following questions where O denotes the origin
(a) Determine vectors $\vec{Q}$ (A to B), $\vec{P}$ (B to C) and $\vec{R}$ (O to A)
(b) What is the distance from A to B?
(c) What is the angle between the vectors $\vec{Q}$ and $\vec{P}$

EP-15. Brackets - Prove that

$$\langle V|W\rangle = \langle W|V\rangle^* \qquad \text{(equation (02\_37)}$$

EP-16. Soft State - Prove $|soft\rangle = |s\rangle = \frac{1}{\sqrt{2}}|g\rangle - \frac{1}{\sqrt{2}}|m\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

EP-17. Operators Acting - Given two operators

$$\hat{O}_1 = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \quad \text{and} \quad \hat{O}_2 = \begin{pmatrix} 10 & 4 \\ -5 & -2 \end{pmatrix}$$

and two vectors

$$|X\rangle = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad \text{and} \quad |Y\rangle = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Determine $\hat{O}_1|X\rangle$, $\hat{O}_2|X\rangle$, $\hat{O}_1|Y\rangle$, and $\hat{O}_2|Y\rangle$

EP-18. Eigenvectors - Here is an operator and four vectors

$$\hat{O} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \; , \; |A\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; |B\rangle = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \; |C\rangle = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \; |D\rangle = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Which of the four vectors are eigenvectors of the operator $\hat{O}$ and what are their eigenvalues?

EP-19. Color Operator - If we know what an operator does, then we can construct it. Let us do this for the color operator. We work in the hardness basis:

$$|hard\rangle = |h\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |soft\rangle = |s\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ with hardness operator } \hat{O}_h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Using this basis, the color states are given by

$$|green\rangle = |g\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |magenta\rangle = |m\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

All we need to remember in order to construct the color operator is how it acts on the color states

$$\hat{O}_c|g\rangle = |g\rangle \quad \text{and} \quad \hat{O}_c|m\rangle = -|m\rangle$$

Write out these last two equations treating the four elements of the color operator as unknowns. You should get four equations that allow you to calculate the four elements of the color operator. Do you get the same operator as in the textbook?

EP-20. Linear Functional - We call $|\psi\rangle$ a ket-vector (state) and $\langle\psi|$ its dual vector or bra-vector (state). Technically the dual vector is a linear functional, that is, a new mathematical object which has the property of turning ket-vectors into numbers (complex). An "operational" way to define the linear functional or bra-vector is as follows.

In a 2-dimensional vector space, if $|\psi\rangle = a_1|1\rangle + a_2|2\rangle$, where $(|1\rangle,|2\rangle)$ form an orthonormal basis, then $\langle\psi| = a_1^*\langle 1| + a_2^*\langle 2|$. It then acts on a ket-vector to produce a scalar. Show that $\langle\psi|\psi\rangle = |a_1|^2 + |a_2|^2$.

EP-22. Bases and Matrices - Suppose we have an orthonormal set of basis vectors $|1\rangle, |2\rangle,$ and $|3\rangle$

(a) Express the orthonormality using scalar products between the basis vectors.

Suppose we have some operator $\hat{G}$ with the following properties with respect to this basis set

$$\hat{G}|1\rangle = 2|1\rangle - 4|2\rangle + 7|3\rangle$$
$$\hat{G}|2\rangle = -2|1\rangle + 3|3\rangle$$
$$\hat{G}|3\rangle = 11|1\rangle + 2|2\rangle - 6|3\rangle$$

(b) Write out the matrix representing the operator $\hat{G}$ in the $\{|1\rangle,|2\rangle,|3\rangle\}$ basis.

EP-23. Eigenvectors - Find the eigenvalues and eigenvectors of the operator

$$\hat{O} = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}$$

EP-24. Spectral Decomposition - Suppose an operator $\hat{K}$ has the following eigenvectors and eigenvalues

$$\hat{K}|1\rangle = 2|1\rangle$$

$$\hat{K}|2\rangle = 3|2\rangle$$

$$\hat{K}|3\rangle = -6|3\rangle$$

(a) Write an expression for $\hat{K}$ in terms of its eigenvalues and eigenvectors (projection operators). Use this expression to derive the matrix representing $\hat{K}$ in the $\{|1\rangle,|2\rangle,|3\rangle\}$ basis.

(b) What is the expectation value of $\hat{K}$ in the state

$$|\alpha\rangle = \frac{1}{\sqrt{83}}\left(-3|1\rangle + 5|2\rangle + 7|3\rangle\right)$$

EP-25. Multiply Matrices

(a) Multiply the two matrices

$$\hat{A} = \begin{pmatrix} 1 & 5 & 4 \\ 7 & 2 & 1 \\ 9 & 2 & 3 \end{pmatrix} \quad , \quad \hat{B} = \begin{pmatrix} 6 & 9 & -2 \\ 5 & 5 & -3 \\ -3 & -5 & 1 \end{pmatrix}$$

(b) Determine the commutator $\left[\hat{A},\hat{B}\right]$

EP-26. Matrix Properties

$$\hat{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{C} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

(a) Show that $\hat{C} = \hat{I} + 2\hat{B}$

(b) Show that $\left[\hat{B},\hat{C}\right] = 0$

(c) Find the eigenvectors and eigenvalues of $\hat{B}$ and $\hat{C}$