BELL'S THEOREM - Introduction

Bell demonstrated that under certain conditions quantum theory and local hidden variable theories **predict different results** for the same experiments on pairs of correlated particles.

This difference, which is intrinsic to **all** local hidden variable theories and is **independent** of the exact nature of the theory, is summarized in Bell's derived inequalities.

This proof **forced** questions about hidden variables to immediately change character.

They were no longer **academic questions** about philosophy but **practical questions** of profound importance for quantum theory.

The choice between quantum theory and local hidden variable theories was no longer matter of **taste**, but matter of **correctness**.

Bertlmann's Socks

Let us derive Bell's theorem with help of a famous Dr Bertlmann (this story by Bell himself). We will do a more mathematical derivation later.

Any philosopher in the street, who has not suffered through a course in quantum mechanics, is quite unimpressed by the Einstein-Podolsky-Rosen correlations.

She can point to many examples of similar correlations in everyday life. The case of **Dr. Bertlmann's socks** is often cited.

Dr. Bertlmann likes to wear two socks of different colors.

Which color he will have on given foot on given day is quite unpredictable. But when you see the first sock is pink you are sure the second sock will not be pink. Observation of first, and experience with Dr. Bertlmann, gives immediate information about second. There is no mystery in such correlations.

Isn’t this EPR business just the same sort of thing?

Dr. Bertlmann happens to be physicist who is very interested in physical characteristics of his socks. He has secured a research grant from a leading sock manufacturer to study how his socks stand up to the rigors of prolonged washing at different temperatures.

Bertlmann decides to subject his left socks (socks A from now on) to 3 different tests:

test a	washing for 1 hour at	0.0 $^\circ C$
test b	washing for 1 hour at	22.5 $^\circ C$
test c	washing for 1 hour at	45.0 $^\circ C$

He is particularly concerned about numbers of socks A that survive intact (a + result) or are destroyed (a – result) by prolonged washing at these different temperatures.

He denotes the number of socks that pass test a and fail test b as

$$n[a+b-]$$

Being a theoretical physicist, he knows that he can discover simple relationships between such numbers without actually performing tests using real socks and real washing machines. This makes his study inexpensive and more attractive to his research sponsors.

He reasons as follows:

$n[a+b-]$ can be written as the sum of the numbers of socks belonging to two subsets, one where individual socks pass test a, fail b and pass c and one where socks pass test a, fail b and fail c, i.e.,

$$n[a+b-] = n[a+b-c+] + n[a+b-c-] \qquad (1)$$

This works because

$$P(C) + P(not\ C) = 1$$

so that equation (1) just says that

$$n[a+b-] = (P[c+] + P[c-])n[a+b-]$$

Similarly, we get

$$n[b+c-] = n[a+b+c-] + n[a-b+c-] \qquad (2)$$

where individual socks pass test b, fail c and pass a and one where socks pass test b, fail c and fail a

and

$$n[a+c-] = n[a+b+c-] + n[a+b-c-] \qquad (3)$$

where individual socks pass test a, fail c and pass b and another one where socks pass test a, fail c and fail b.

From equation (1) it follows that

$$n[a+b-] \geq n[a+b-c-] \qquad (4)$$

since all the numbers involved are ≥ 0.

From equation (2) it follows that

$$n[b+c-] \geq n[a+b+c-] \qquad (5)$$

Adding equations (4) and (5) gives the result

$$n[a+b-] + n[b+c-] \geq n[a+b-c-] + n[a+b+c-] = n[a+c-]$$

or

$$n[a+b-]+n[b+c-]\geq n[a+c-] \quad (6)$$

At this stage, Dr. Bertlmann notices flaw in reasoning, which all of you will, **of course**, have spotted right at the beginning.

Subjecting one of socks A to test a will necessarily change irreversibly its physical characteristics such that, even if it survives the test, it may not give the result for test b that might be expected of brand new sock.

And, of course, if sock fails test b , it will simply not be available(destroyed) for test c .

The numbers $n[a+b-]$, etc, therefore have no practical (we cannot measure them in real world) relevance.

Bertlmann now remembers his socks always come in pairs.

He assumes that, apart from differences in the color, the physical characteristics of each sock in a pair are identical.

Thus, a test performed on the right sock (sock B) can be used to predict what the result of same test would be if the test had been performed on left sock(sock A), even though the test on A is **not actually carried out**.

He must further assumes that whatever test he chooses to perform on B in no way affects the outcome of any other test he might perform on A, **but this seems so obviously valid that he does not give it any thought whatsoever.** Can you figure out where we are in the EPR argument?

Bertlmann now devises three different sets of experiments to be carried out on three samples each containing the **same total number** of pairs of socks.

In experiment 1, for each pair, sock A is subjected to test a and sock B is subjected to test b . If sock B fails test b, this implies that sock A would also have failed test b had it been performed on sock A.

Thus, the number of pairs of socks for which sock A passes test a and sock B fails test b , which we denote by

$$N_{+-}(a,b)$$

must be equal to the (hypothetical) number of socks A which pass test a **and** fail test b , i.e.,

$$N_{+-}(a,b)=n[a+b-] \quad (7)$$

In experiment 2, for each pair, sock A is subjected to test b and sock B is subjected to test c . The same kind of reasoning allows Bertlmann to deduce that

$$N_{+-}(b,c)=n[b+c-] \quad (8)$$

where $N_{+-}(b,c)$ denotes the number of pairs of socks for which sock A passes test b and sock B fails test c .

Finally, in experiment 3, for each pair, sock A is subjected to test a and sock B is subjected to test c . In a similar manner, Bertlmann deduces that

$$N_{+-}(a,c) = n[a + c -] \qquad (9)$$

where $N_{+-}(a,c)$ denotes the number of pairs of socks for which sock A passes test a and sock B fails test c .

The experimental arrangements are summarized below.

Experiment	*Sock A Test*	*Sock B Test*
1	*a*	*b*
2	*b*	*c*
3	*a*	*c*

Using eqs. (7)-(9) and (6) Bertlmann then concludes that we must have the inequality

$$N_{+-}(a,b) + N_{+-}(b,c) \geq N_{+-}(a,c) \qquad (10)$$

Bertlmann generalizes this result for any batch of pairs of socks by dividing each number in equation (10) by the total number of pairs of socks (which was same for each experiment) to arrive at frequencies with which each joint result was obtained.

He identifies these frequencies with probabilities for obtaining results for experiments to be performed on any batch of pairs of socks that, statistically, have same properties.

Thus, he finds that

$$P_{+-}(a,b) + P_{+-}(b,c) \geq P_{+-}(a,c) \qquad (11)$$

This is **Bell's inequality** for this experiment.

Now we follow the above arguments again, replacing

socks with photons

pairs of socks with pairs of entangled photons

washing machines with polarization analyzers and

temperatures with polarizer orientations

and we will still arrive at Bell's inequality, equation (11), i.e., we only change the words in our description of the experiments and not the conclusions!

Our three tests now refer to polarization analyzers set with their vertical(optic) axes oriented at

$$a \to 0.0°$$
$$b \to 22.5°$$
$$c \to 45.0°$$

The different experimental arrangements are summarized as follows:

Expt	Photon A Orientation	Photon B Orientation	*Difference*
1	$a = 0.0°$	$b = 22.5°$	$b - a = 22.5°$
2	$b = 22.5°$	$c = 45.0°$	$c - b = 22.5°$
3	$a = 0.0°$	$c = 45.0°$	$c - a = 45.0°$

The probabilities predicted by quantum theory for difference angle $(b - a)$ between the polaroids in each test are given by

$$\frac{1}{2}\sin^2(b - a)$$

i.e., remember the matches and misses arguments.

Putting the angles above into equation (11) we get the Bell inequality

$$\frac{1}{2}\sin^2 22.5° + \frac{1}{2}\sin^2 22.5° \geq \frac{1}{2}\sin^2 45.0° \qquad (12)$$

or

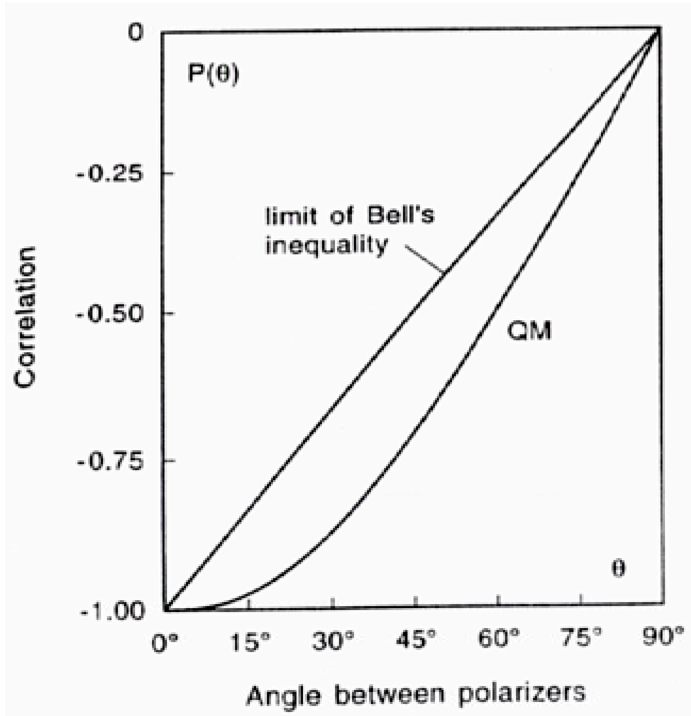
$$0.1464 \geq 0.2500 \qquad (13)$$

which is **obviously incorrect**!

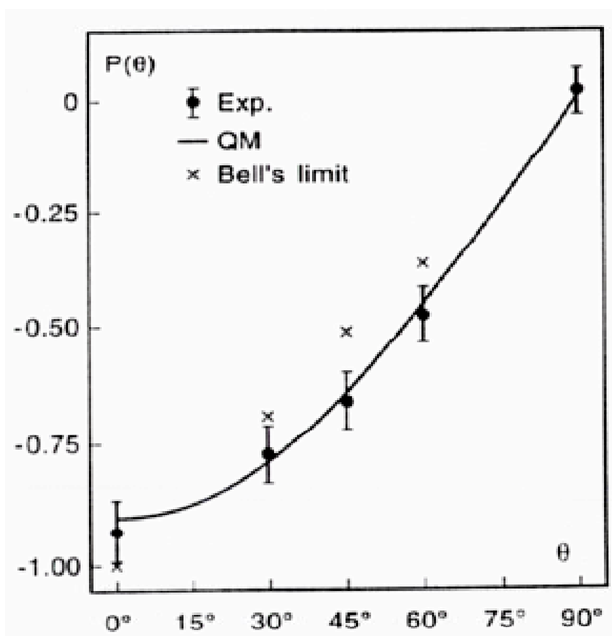
Thus, for these particular arrangements of polarization analyzers, the probability formula from quantum theory predicts results that violate Bell's inequality.

However, the quantum mechanical probability formula results agrees with experimental observations!!

The comparison of quantum mechanics and the Bell inequality is shown in the figure below:



The experimental demonstration of the violation of the Bell inequality on the spin correlation of proton pairs is shown below.



Clearly, quantum mechanics is correct.

The most important assumption made in the reasoning which led to the inequality was Einstein separability or local reality of the photons(or socks).

The inequality is quite independent of nature of any local hidden variable theory that could be devised.

The conclusion is inescapable.

Quantum theory is incompatible with any local hidden variable theory and hence incompatible with any form of local reality.

This means that any theory that has the same probabilistic predictions as quantum mechanics must be nonlocal.

We should not, perhaps, be too surprised by this result.

The predictions of quantum theory are based on the properties of a 2-particle state vector which, before collapsing into one of **measurement eigenstates**, is **delocalized or entangled** over the whole experimental arrangement.

The 2 particles are, in effect, always in "**contact"** prior to measurement and can therefore exhibit a degree of correlation impossible for 2 Einstein separable (locally realistic) particles.

Bell's inequality provides us with a straightforward test. If experiments like those described here are actually performed, the results allow us to make a choice between quantum theory and a whole range of theories based on local hidden variables and hidden variable theories are ruled out conclusively.

Bohr once declared when he was asked whether the quantum mechanical algorithms could be considered as somehow mirroring an underlying quantum reality: He said :

> There is no quantum world.
>
> There is only an abstract quantum mechanical description.
>
> It is wrong to think that the task of physics is to find out how Nature is.
>
> Physics is concerned only with what we **can** say about Nature.

Heisenberg said:

> In the experiments about atomic events we have to do with things and facts, with phenomena that are just as real as any phenomena in daily life. But the atoms or the elementary particles are not as real; they form a world of **potentialities or possibilities** rather than one of real things or facts.

Jordan declared:

> That observations not only **disturb** what

> has to be measured, they **produce** it. In a measurement of position of an electron, the electron is forced to a decision.
>
> We compel it to **assume a definite position;** previously it was, in general, neither here nor there; it had not yet made its decision about a definite position.

Next, we will do a more mathematically rigorous derivation of the EPR/Bell phenomena.

Then we will move on to a discussion of this new **quantum reality**.