The Strange World of Neutral K-Mesons

We can now use the same formalism we developed for photon polarization to study elementary particles called K-mesons.

K-mesons are produced in high-energy accelerators via the **production** process

$$\pi^- + p^+ \rightarrow \Lambda^0 + K^0$$

In this reaction, electric charge is conserved. This reaction takes place via the so-called strong interactions. Another physical quantity called **strangeness** is also conserved in strong interactions. All K^0-mesons have a strangeness equal to +1.

For every particle there always exists an antiparticle. For the K^0, the antiparticle is called the $\overline{K}^0$. The $\overline{K}^0$-mesons have a strangeness equal to -1. A reaction involving the $\overline{K}^0$ is

$$\overline{K}^0 + p^+ \rightarrow \Lambda^0 + \pi^+$$

which is an **absorption** process.

The K-mesons that exist in the experimental world(the laboratory) are linear superpositions of K^0 and $\overline{K}^0$ states in the same way that RCP and LCP photons were superpositions of the $|x\rangle$ and $|y\rangle$ polarization states. So the world of K-mesons can be represented by a 2-dimensional vector space.

One basis for the vector space is the orthonormal set $\left\{|K^0\rangle, |\overline{K}^0\rangle\right\}$ where

$$\langle K^0 | K^0 \rangle = 1 = \langle \overline{K}^0 | \overline{K}^0 \rangle \quad \text{and} \quad \langle K^0 | \overline{K}^0 \rangle = 0$$

Two linear operators are important for the study of K-mesons.

First, we represent the strangeness operator. We already stated that the states $|K^0\rangle$ and $|\overline{K}^0\rangle$ have definite values of strangeness, which means that they are eigenvectors of the strangeness operator $\hat{S}$ with eigenvalues ± 1(by convention). Using our formalism, this means that

$$\hat{S}|K^0\rangle = |K^0\rangle \quad \text{and} \quad \hat{S}|\overline{K}^0\rangle = -|\overline{K}^0\rangle$$

$$\hat{S} = \begin{pmatrix} \langle K^0|\hat{S}|K^0\rangle & \langle K^0|\hat{S}|\overline{K}^0\rangle \\ \langle \overline{K}^0|\hat{S}|K^0\rangle & \langle \overline{K}^0|\hat{S}|\overline{K}^0\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |K^0\rangle\langle K^0| - |\overline{K}^0\rangle\langle \overline{K}^0|$$

in the $\left\{|K^0\rangle, |\overline{K}^0\rangle\right\}$ basis.

The second linear operator that is important in the K-meson system is **charge conjugation** $\hat{C}$. This operator changes particles into antiparticles and vice versa. In the K-meson system using the $\left\{|K^0\rangle, |\overline{K}^0\rangle\right\}$ basis we **define** $\hat{C}$ by the particle-antiparticle changing relations

$$\hat{C}\left|K^0\right\rangle = \left|\bar{K}^0\right\rangle \quad \text{and} \quad \hat{C}\left|\bar{K}^0\right\rangle = \left|K^0\right\rangle$$

$$\hat{C} = \begin{pmatrix} \left\langle K^0\middle|\hat{C}\middle|K^0\right\rangle & \left\langle K^0\middle|\hat{C}\middle|\bar{K}^0\right\rangle \\ \left\langle \bar{K}^0\middle|\hat{C}\middle|K^0\right\rangle & \left\langle \bar{K}^0\middle|\hat{C}\middle|\bar{K}^0\right\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We can find the eigenvectors and eigenvalues of the $\hat{C}$ operator as follows

$$\hat{C}|\psi\rangle = \lambda|\psi\rangle$$

$$\hat{C}^2|\psi\rangle = \lambda\hat{C}|\psi\rangle = \lambda^2|\psi\rangle = \hat{I}|\psi\rangle = |\psi\rangle$$

where we have used $\hat{C}^2 = \hat{I}$. This says that $\lambda^2 = 1$ or the eigenvalues of $\hat{C}$ are ± 1. If we use the $\left\{\left|K^0\right\rangle, \left|\bar{K}^0\right\rangle\right\}$ basis and assume that

$$|\psi\rangle = a\left|K^0\right\rangle + b\left|\bar{K}^0\right\rangle \quad \text{where} \quad |a|^2 + |b|^2 = 1$$

we find for $\lambda = +1$

$$\hat{C}|\psi\rangle = a\hat{C}\left|K^0\right\rangle + b\hat{C}\left|\bar{K}^0\right\rangle = |\psi\rangle = a\left|K^0\right\rangle + b\left|\bar{K}^0\right\rangle$$

$$a\left|\bar{K}^0\right\rangle + b\left|K^0\right\rangle = a\left|K^0\right\rangle + b\left|\bar{K}^0\right\rangle$$

or $a = b = \frac{1}{\sqrt{2}}$. If we define the $+1$ eigenvector as $|K_S\rangle$, we then have

$$|K_S\rangle = \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle + \left|\bar{K}^0\right\rangle\right)$$

Similarly, if we define the -1 eigenvector as $|K_L\rangle$, we then have

$$|K_L\rangle = \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle - \left|\bar{K}^0\right\rangle\right)$$

We then have

$$\hat{C}|K_S\rangle = |K_S\rangle \quad \text{and} \quad \hat{C}|K_L\rangle = -|K_L\rangle$$

Since the commutator $\left[\hat{S},\hat{C}\right] \neq 0$, these two operators do not have a common set of eigenvectors. This means that both operators cannot have definite values in the same state. In fact, the concept of charge conjugation is meaningless for K-mesons in the $\left\{\left|K^0\right\rangle, \left|\bar{K}^0\right\rangle\right\}$ states and the concept of strangeness is meaningless for K-mesons in the $\left\{|K_S\rangle, |K_L\rangle\right\}$ states.

The $\left\{|K_S\rangle, |K_L\rangle\right\}$ states form a second orthonormal basis for the vector space (like the RCP and LCP polarization states).

The **standard approach** we follow when studying physical systems using quantum mechanics is

(1) define the Hamiltonian for the system
(2) find its eigenvalues and eigenvectors
(3) investigate the time development operator generated by the Hamiltonian and

(4) calculate transition probabilities connected to experiments

Along the way we will define the properties of other operators appropriate to the system under investigation (like $\hat{S}$ and $\hat{C}$ above).

It is the job of the theoretical physicist to **derive or guess** an appropriate Hamiltonian.

Since we are in 2-dimensional vector space, all operators are represented by 2x2 matrices. In the case of the K-meson system, we will assume the most general form constructed from all of the relevant operators. Therefore, we assume

$$\hat{H} = M\hat{I} + A\hat{C} + B\hat{S} = \begin{pmatrix} M+B & A \\ A & M-B \end{pmatrix}$$

and investigate the consequences of this assumption. We will assume that the matrix has been written down in the $\left\{ \left|K^0\right\rangle, \left|\bar{K}^0\right\rangle \right\}$ basis.

Step 1

Investigate the commutators:

$$\left[\hat{H},\hat{H}\right] = 0 \ , \ \left[\hat{H},\hat{S}\right] \neq 0 \ , \ \left[\hat{H},\hat{C}\right] \neq 0 \ , \ \left[\hat{S},\hat{C}\right] \neq 0$$

If an operator commutes with the Hamiltonian, then the corresponding physical quantity is a constant of the motion.

Since the Hamiltonian always commutes with itself and it is not explicitly dependent on time, the physical observable connected to the Hamiltonian, namely the energy, is conserved.

Since they do not commute with the assumed form of $\hat{H}$, neither $\hat{S}$ nor $\hat{C}$ is conserved in this model.

When a physical observable is conserved, we say its value corresponds to a good **quantum number** that can be used to characterize(label) the ket vector representing the physical system.

Step 2

Investigate **special cases** (limits of the most general solution):

Case of A = 0 :

$$\hat{H} = M\hat{I} + B\hat{S} = \begin{pmatrix} M+B & 0 \\ 0 & M-B \end{pmatrix}$$

Now

$$\left[\hat{H},\hat{S}\right] = 0$$

which means that $\hat{H}$ and $\hat{S}$ share a common set of eigenvectors and $\hat{S}$ is a constant of the motion. We already know the eigenvectors or $\hat{S}$ and so the eigenvector/eigenvalue problem for $\hat{H}$ is already solved(**clearly**

this is a very powerful rule). We have

$$\hat{H}\left|K_o\right\rangle = (M+B)\left|K_o\right\rangle \text{ and } \hat{H}\left|\bar{K}_o\right\rangle = (M-B)\left|\bar{K}_o\right\rangle$$

We could have surmised this from the diagonal form of the matrix representation, since the only way the matrix could be diagonal is for the basis states of the representation to be its eigenvectors.

$\hat{C}$ is not conserved in this case, since $\left[\hat{H},\hat{C}\right] \neq 0$.

The energy eigenstates, $\left\{\left|K^0\right\rangle, \left|\bar{K}^0\right\rangle\right\}$ in this case, are a basis for the vector space. This means that we can write any arbitrary vector as a linear combination of these vectors

$$\left|\psi\right\rangle = a\left|K_0\right\rangle + b\left|\bar{K}_0\right\rangle = a\left|E = M+B\right\rangle + b\left|E = M-B\right\rangle$$

Now, as we derived earlier, energy eigenstates have a simple time dependence

$$\hat{H}\left|E\right\rangle = E\left|E\right\rangle$$

$$\hat{U}(t)\left|E\right\rangle = e^{-i\frac{\hat{H}}{\hbar}t}\left|E\right\rangle = e^{-i\frac{E}{\hbar}t}\left|E\right\rangle$$

Therefore, in this case, the time dependence of the arbitrary state vector is given by

$$\left|\psi(t)\right\rangle = ae^{-i\frac{M+B}{\hbar}t}\left|K_0\right\rangle + be^{-i\frac{M-B}{\hbar}t}\left|\bar{K}_0\right\rangle$$

This will be a general approach we will use, i.e., expand an arbitrary state in energy eigenstates and use the simple time dependence of the energy eigenstates to determine the more complex time dependence of the arbitrary state. Of course, we have to be able to solve the eigenvector/eigenvalue problem for the Hamiltonian(the energy operator) of the system under investigation.

Case of B = 0 :

$$\hat{H} = M\hat{I} + A\hat{C} = \begin{pmatrix} M & A \\ A & M \end{pmatrix}$$

Now

$$\left[\hat{H},\hat{C}\right] = 0$$

which means that $\hat{H}$ and $\hat{C}$ share a common set of eigenvectors and $\hat{C}$ is a constant of the motion. We already know the eigenvectors of $\hat{C}$ and so the eigenvector/eigenvalue problem for $\hat{H}$ is again already solved. We have

$$\hat{H}\left|K_S\right\rangle = (M+A)\left|K_S\right\rangle \text{ and } \hat{H}\left|K_L\right\rangle = (M-A)\left|K_L\right\rangle$$

$\hat{S}$ is not conserved in this case, since $\left[\hat{H},\hat{S}\right] \neq 0$.

The energy eigenstates, $\left\{\left|K_S\right\rangle, \left|K_L\right\rangle\right\}$ in this case, are a basis for the vector space. In this basis

$$\hat{H} = \begin{pmatrix} M+A & 0 \\ 0 & M-A \end{pmatrix}$$

as expected.

We could also solve this problem by finding the characteristic equation for the Hamiltonian matrix, i.e., since we have

$$\hat{H}|\psi\rangle = \lambda|\psi\rangle \to (\hat{H} - \lambda\hat{I})|\psi\rangle = 0$$

the characteristic equation is

$$\det(\hat{H} - \lambda\hat{I}) = 0 = \det\begin{pmatrix} M-\lambda & A \\ A & M-\lambda \end{pmatrix}$$

$$\left(M-\lambda\right)^2 - A^2 = 0 \to \lambda = M \pm A$$

Since we have another basis, we can write any arbitrary vector as a linear combination of these vectors

$$|\psi\rangle = a|K_S\rangle + b|K_L\rangle = a|E = M + A\rangle + b|E = M - A\rangle$$

Therefore, in this case, we have the time dependence

$$|\psi(t)\rangle = ae^{-i\frac{M+A}{\hbar}t}|K_S\rangle + be^{-i\frac{M-A}{\hbar}t}|K_L\rangle$$

Step 3

Solve the general Hamiltonian problem (if possible; otherwise we must use approximation methods). We have

$$\hat{H} = M\hat{I} + A\hat{C} + B\hat{S} = \begin{pmatrix} M+B & A \\ A & M-B \end{pmatrix}$$

We assume that the eigenvectors satisfy $\hat{H}|\phi\rangle = E|\phi\rangle$ where

$$|\phi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

This gives

$$\begin{pmatrix} M+B & A \\ A & M-B \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = E\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

or

$$\begin{pmatrix} M+B-E & A \\ A & M-B-E \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0$$

This is a set of two homogeneous equations in two unknowns. It has a nontrivial solution only if the determinant of the coefficients is zero

$$\begin{vmatrix} M+B-E & A \\ A & M-B-E \end{vmatrix} = 0 = (M+B-E)(M-B-E) - A^2$$

This has solution

$$E_{\pm} = M \pm \sqrt{A^2 + B^2} \qquad \text{(the energy eigenvalues)}$$

We solve for the eigenstates by substituting the eigenvalues into the eigenvalue/eigenvector equation

$$\begin{pmatrix} M+B & A \\ A & M-B \end{pmatrix} \begin{pmatrix} \phi_{1\pm} \\ \phi_{2\pm} \end{pmatrix} = E_{\pm} \begin{pmatrix} \phi_{1\pm} \\ \phi_{2\pm} \end{pmatrix} = E_{\pm} \left| \phi_{\pm} \right\rangle$$

After some algebra we get

$$\frac{\phi_{1\pm}}{\phi_{2\pm}} = \frac{-A}{B \mp \sqrt{A^2 + B^2}} = \frac{B \pm \sqrt{A^2 + B^2}}{A}$$

We check the validity of this solution by comparing it the limiting cases

For B = 0, we have

$$\frac{\phi_{1\pm}}{\phi_{2\pm}} = \frac{-A}{\mp A} = \pm 1 \rightarrow \phi_{1+} = \phi_{2+} \quad \text{and} \quad \phi_{1-} = -\phi_{2-}$$

which says that

$$\left| \phi_{+} \right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \left| K_S \right\rangle \quad \text{and} \quad \left| \phi_{-} \right\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left| K_L \right\rangle$$

which agrees with the earlier results.

In the other limiting case, A = 0, we have

$$\frac{\phi_{1+}}{\phi_{2+}} = \infty \rightarrow \phi_{1+} = 1 \quad \text{and} \quad \phi_{2+} = 0$$

$$\frac{\phi_{1-}}{\phi_{2-}} = 0 \rightarrow \phi_{1-} = 0 \quad \text{and} \quad \phi_{2-} = 1$$

which says that

$$\left| \phi_{+} \right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left| K_0 \right\rangle \quad \text{and} \quad \left| \phi_{-} \right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left| \overline{K}_0 \right\rangle$$

which again agrees with the earlier results.

If we normalize the general solution

$$\frac{\phi_{1\pm}}{\phi_{2\pm}} = \frac{-A}{B \mp \sqrt{A^2 + B^2}} = \frac{B \pm \sqrt{A^2 + B^2}}{A}$$

using

$$\left|\phi_{1\pm}\right|^2 + \left|\phi_{2\pm}\right|^2 = 1$$

we obtain

$$\phi_{1\pm} = \frac{A}{\sqrt{A^2 + (B \mp \sqrt{A^2 + B^2})^2}}$$

$$\phi_{2\pm} = \frac{B \mp \sqrt{A^2 + B^2}}{\sqrt{A^2 + (B \mp \sqrt{A^2 + B^2})^2}}$$

and

$$\begin{aligned} \left|\phi_\pm\right\rangle &= \frac{1}{\sqrt{A^2 + (B \mp \sqrt{A^2 + B^2})^2}} \begin{pmatrix} A \\ -B \pm \sqrt{A^2 + B^2} \end{pmatrix} \\ &= \frac{1}{\sqrt{A^2 + (B \mp \sqrt{A^2 + B^2})^2}} \left[A\left|K_0\right\rangle + (-B \pm \sqrt{A^2 + B^2})\left|\bar{K}_0\right\rangle \right] \\ &= \frac{1}{2\sqrt{A^2 + (B \mp \sqrt{A^2 + B^2})^2}} \left[(A - B \pm \sqrt{A^2 + B^2})\left|K_S\right\rangle + (A + B \pm \sqrt{A^2 + B^2})\left|K_L\right\rangle \right] \end{aligned}$$

Step 4

Look at a realistic physical system that we can relate to experiment.

In the real world of K-mesons, the Hamiltonian is such that $B << A$. In this case the states $\left\{\left|K_S\right\rangle, \left|K_L\right\rangle\right\}$ are **almost** energy eigenstates or charge conjugation is **almost** conserved. We expect that instead of being able to write

$$\left|\phi_+\right\rangle = \left|K_S\right\rangle \quad \text{and} \quad \left|\phi_-\right\rangle = \left|K_L\right\rangle$$

which would be true if B = 0, that we should be able to write

$$\left|\phi_+\right\rangle = \cos\frac{\theta}{2}\left|K_S\right\rangle + \sin\frac{\theta}{2}\left|K_L\right\rangle$$

$$\left|\phi_-\right\rangle = -\sin\frac{\theta}{2}\left|K_S\right\rangle + \cos\frac{\theta}{2}\left|K_L\right\rangle$$

where for $\theta << 1$ we clearly approximate the B = 0 result. Let us see how this works. For $B << A$, we choose

$$\frac{\theta}{2} = \frac{B}{2A} << 1$$

and get

$$\frac{\theta}{2} \approx \tan\frac{\theta}{2} = \frac{B}{2A} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

To lowest order we can say

$$\sin\frac{\theta}{2} = \frac{B}{2A} = \delta << 1 \quad \text{and} \quad \cos\frac{\theta}{2} = 1$$

to get

$$|\phi_+\rangle = |K_S\rangle + \delta|K_L\rangle$$
$$|\phi_-\rangle = |K_L\rangle - \delta|K_S\rangle$$

This says that if $|\psi_{in}\rangle = |\phi_-\rangle$, then the number

$$\frac{|\langle K_S|\phi_-\rangle|^2}{|\langle K_L|\phi_-\rangle|^2} = \frac{\text{probability of observing a } K_S}{\text{probability of observing a } K_L}$$
$$= \frac{(\text{Number of } K_S)/(\text{Number of } K_S \text{ and } K_L\ )}{(\text{Number of } K_L)/(\text{Number of } K_S \text{ and } K_L\ )}$$
$$= \frac{(\text{Number of } K_S)}{(\text{Number of } K_L)}$$

gives the experimental ratio of the number of times we will measure a final state of $|K_S\rangle$ to the number of times we will measure the final state of $|K_L\rangle$. The signature for seeing a final state of $|K_L\rangle$ is to see its decay to 3 π-mesons and that of a final state of $|K_S\rangle$ is to see its decay to 2 π-mesons. The number is

$$\frac{|\langle K_S|\phi_-\rangle|^2}{|\langle K_L|\phi_-\rangle|^2} = |\delta|^2$$

Now experiment gives the result $|\delta| = 2x10^{-3}$. This number is a measure of how large of an effect strangeness has on this system.

If B = 0, then charge conjugation is conserved. If B ≠ 0, then charge conjugation is not conserved. So δ is measure of the lack of charge conjugation conservation in the K-meson system

If we identify the energy eigenvalues as the particle **rest energies**

$$M + A = m_S c^2$$
$$M - A = m_L c^2$$

we have

$$A = \frac{m_S c^2 - m_L c^2}{2} = 10^{-5}\text{ eV}$$

and

$$B = 2A\delta = 10^{-17} m_K c^2$$

or

$$\frac{B}{m_K c^2} = 10^{-17}$$

This is a one part in 10^{17} effect!!! It is one of the best measurements ever made. It won the Nobel prize for the experimenters. It is now understood in detail by the standard model of elementary particles.

Now let us look at **Quantum Interference Effects** in this K-meson system.

Suppose that B = 0. Then the energy eigenstates are $\left\{\left|K_S\right\rangle,\left|K_L\right\rangle\right\}$ with eigenvalues $M \pm A$. Now let $\left|\psi_{in}\right\rangle = \left|K_0\right\rangle = \frac{1}{\sqrt{2}}(\left|K_S\right\rangle + \left|K_L\right\rangle)$. This is **not an energy eigenstate so it will evolve in time(components will change their relative phase)**. Its time evolution is given by

$$\begin{aligned}\left|\psi(t)\right\rangle &= e^{-i\frac{\hat{H}}{\hbar}t}\left|\psi_{in}\right\rangle = e^{-i\frac{\hat{H}}{\hbar}t}\left|K_0\right\rangle \\ &= \frac{1}{\sqrt{2}}(e^{-i\frac{\hat{H}}{\hbar}t}\left|K_S\right\rangle + e^{-i\frac{\hat{H}}{\hbar}t}\left|K_L\right\rangle) \\ &= \frac{1}{\sqrt{2}}(e^{-i\frac{M+A}{\hbar}t}\left|K_S\right\rangle + e^{-i\frac{M-A}{\hbar}t}\left|K_L\right\rangle)\end{aligned}$$

The probability amplitude that the initial meson **oscillates** into the orthogonal state $\overline{K}_0$ at time t is given by

$$\begin{aligned}\left\langle\overline{K}_0\middle|\psi(t)\right\rangle &= \frac{1}{\sqrt{2}}(e^{-i\frac{M+A}{\hbar}t}\left\langle\overline{K}_0\middle|K_S\right\rangle + e^{-i\frac{M-A}{\hbar}t}\left\langle\overline{K}_0\middle|K_L\right\rangle) \\ &= \frac{1}{\sqrt{2}}(e^{-i\frac{M+A}{\hbar}t}\frac{1}{\sqrt{2}} - e^{-i\frac{M-A}{\hbar}t}\frac{1}{\sqrt{2}})\end{aligned}$$

Finally, the probability that the incoming K_0-meson will behave like a $\overline{K}_0$-meson at time t (that it has oscillated into a $\overline{K}_0$) is given by

$$P_{\overline{K}_0}(t) = \left|\left\langle\overline{K}_0\middle|\psi(t)\right\rangle\right|^2 = \frac{1}{2}\left[1-\cos\Omega t\right]$$

where

$$\Omega = \frac{1}{\hbar}((M+A)-(M-A)) = \frac{2A}{\hbar} = \frac{m_S c^2 - m_L c^2}{\hbar}$$

What is the physics here? If $m_S - m_L = 0$ then $\Omega = 0$ and $P_{\overline{K}_0}(t) = 0$ or the two mesons do not **change** into one another(called **oscillation**) as time passes.

However, if $m_S - m_L \neq 0$ then $P_{\overline{K}_0}(t) \neq 0$ and the two mesons oscillate back and forth, sometimes being a K_0 and sometimes being a $\overline{K}_0$. This has been observed in the laboratory and is, in fact, the way that the extremely small mass difference is actually measured.

This is also the same mechanism that is proposed for oscillations between the different flavors of neutrinos. Experiment indicates that neutrino oscillation is taking place and the explanations is that all the neutrino masses are not be zero!